Methoden der stochastischen Analyse

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Weimarer Optimierungs- und Stochastiktage 1.0
2./3. Dezember 2004

Outline

- Introduction
- Probability and Statistics
- Optimization with stochastic parameters
- Robustness of design
- Reliability-based optimization
- Concluding remarks

Introduction

- Structural Models become increasingly detailed
- Numerical procedures become more and more complex
- Substantially more precise data is required for the analysis
- Optimized designs lead to high imperfection sensitivities

➪ Consideration of random uncertainties becomes mandatory

Probability-based analysis

- Analysis of variances (global variability, robustness)
  - Perturbation approach, Taylor series expansion
  - Plain Monte Carlo simulation, Latin Hypercube sampling
- Safety analysis (extremely rare events, reliability-based optimization)
  - Approximations (response surface method, first order reliability method)
  - Exact solutions (integration, advanced Monte Carlo simulation)

Statistical Characterization of random variables

- Mean value and standard deviation

$$\bar{X} = E[X]; \quad \sigma_X = \sqrt{E[(X-\bar{X})^2]}$$

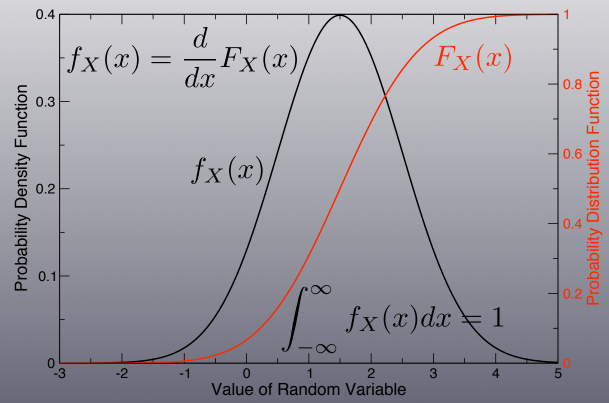
- Coefficient of variation of one variable

$$cov = \frac{\sigma_X}{\bar{X}}$$

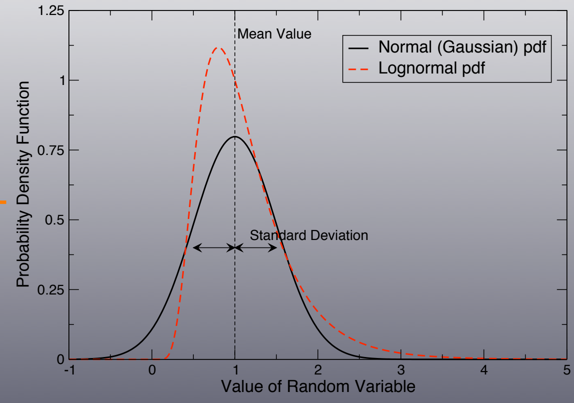
- Coefficient of correlation between two variables

$$\rho_{12} = \frac{E[X_1 X_2]}{\sigma_{X_1}\sigma_{X_2}}$$

Probability Distribution and Probability Density Functions



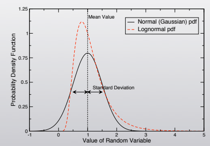
Normal and Lognormal Distribution



Mean value = 1
COV = 50%

"Six Sigma"

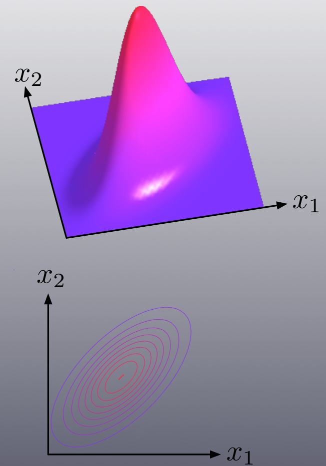
- Representation of "rare" events



Distribution Type	Threshold level	Exceedance Probability
Normal	6 σ	1 x 10^{-9}
Lognormal	6 σ	0.000224
Unknown (Chebyshev bound)	6 σ	0.0278
Lognormal	16 σ	1 x 10^{-9}

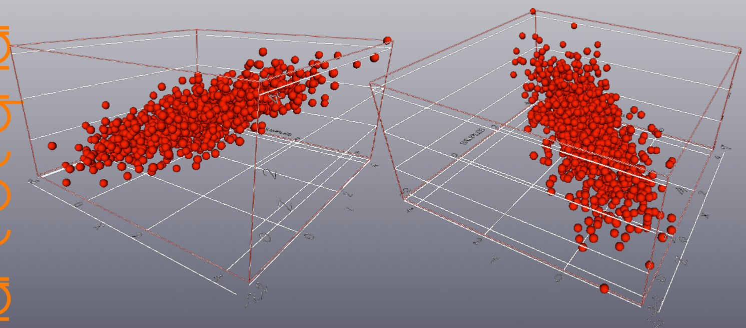
Joint probability density

- Depends on all random variables involved
- Prescribed marginal distribution
- Prescribed coefficient of correlation



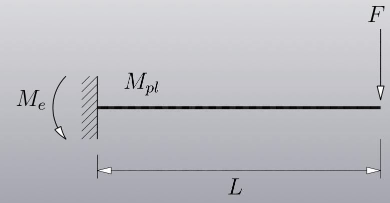
Statistics and Estimation

- Samples of three correlated random variables



Failure probability

- Failure condition is defined in terms of a (non-unique) limit state function
- Failure set and probability are unique

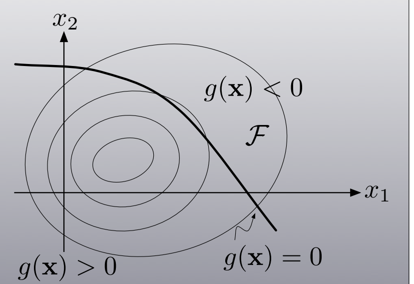


$$\mathcal{F} = \{(F, L, M_{pl}) : FL \geq M_{pl}\}$$
$$= \{(F, L, M_{pl}) : 1 - \frac{FL}{M_{pl}} \leq 0\}$$

$$P(\mathcal{F}) = P[\{\mathbf{X} : g(\mathbf{X}) \leq 0\}]$$

Calculation of failure probability

- Integrate the joint probability density function of the basic variables over the failure domain
- Monte Carlo is ineffective

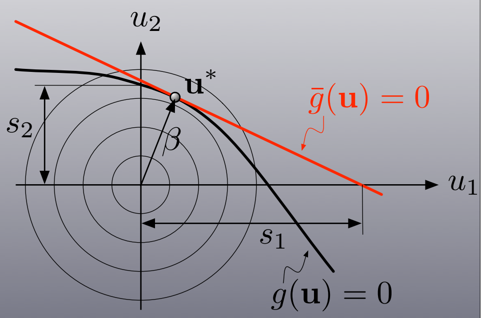


$$P(\mathcal{F}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} I_g(x) f_{X_1 \dots X_n} dx_1 \dots dx_n$$

$$I_g(x_1 \dots x_n) = \begin{cases} 1 : & g(x_1 \dots x_n) \leq 0 \\ 0 : & \text{else} \end{cases}$$

First order reliability method FORM

- Transform to uncorrelated Gaussian space
- Find design point with minimum distance from origin
- Linearize at design point



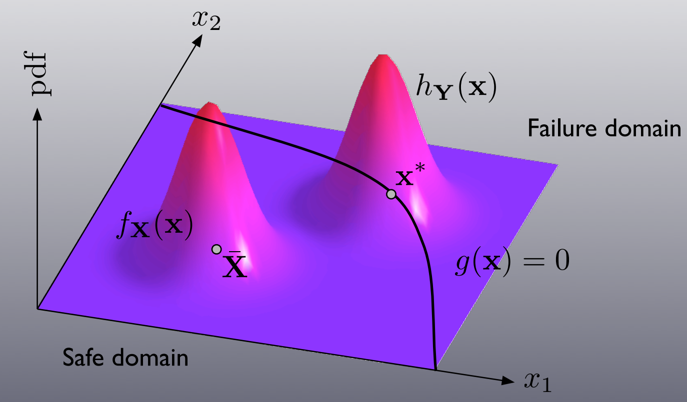
$$P(\mathcal{F}) = \Phi(-\beta)$$

Importance Sampling

- Concentrate samples in the failure region
- Compensate for unlikeliness by weighting
- Obtain an unbiased estimator for failure probability with reduced estimation error

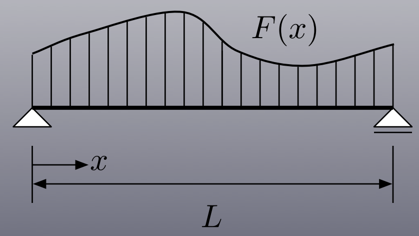
$$\bar{P}(\mathcal{F}) = \frac{1}{m}\sum_{k=1}^{m}\frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})}I_g(\mathbf{x}) = E[\frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})}I_g(\mathbf{x})]$$

Importance Sampling



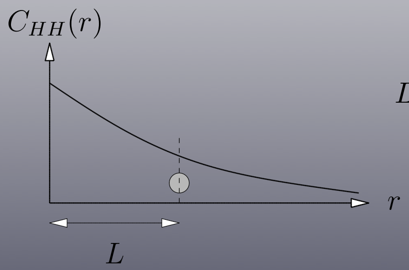
Random fields

- Probabilistic properties depend on continuous spatial variable(s)
- Spatial discretization required for stochastic finite element analysis



Correlation length

- Measure of "waviness" of random fields
- Infinite correlation length reduced random field to simple random variable



$$L = \frac{\int_0^\infty r |C_{HH}(r)| dr}{\int_0^\infty |C_{HH}(r)| dr}$$

Spectral decomposition

- Representation of the random field in terms of:
  - Orthogonal spatial shape functions $\phi_k(\mathbf{x_i})$
  - independent random variables c_k

$$C_{FF}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{\infty} \lambda_k \phi_k(\mathbf{x}) \phi_k(\mathbf{y})$$

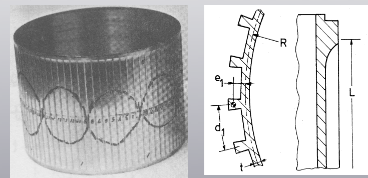
$$F_i = \sum_{k=1}^{\infty} \phi_k(\mathbf{x_i}) c_k = \sum_{k=1}^{\infty} \phi_{ik} c_k$$

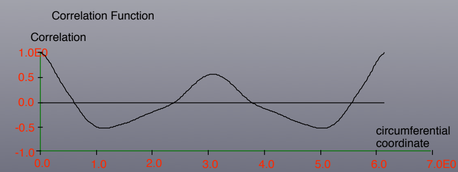
Static stability of shells with random imperfections

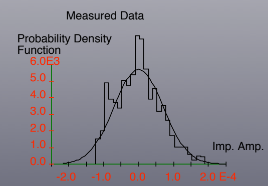
- Random geometrical imperfections
- Random field estimation from measured data
- Spectral decomposition of spatial randomness
- Geometrically nonlinear bifurcation analysis
- Reliability estimation

Measured imperfections

- Singer, Arbocz, Babcock 1971

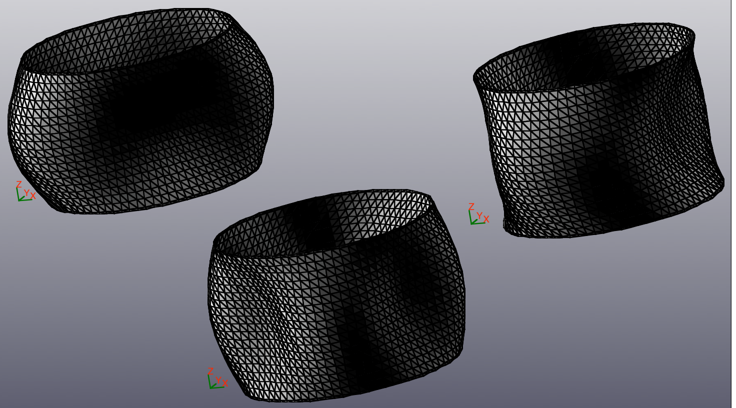




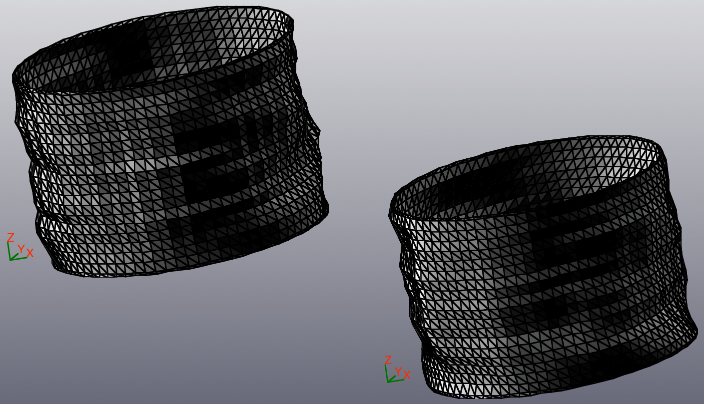


Eigenvectors of random field

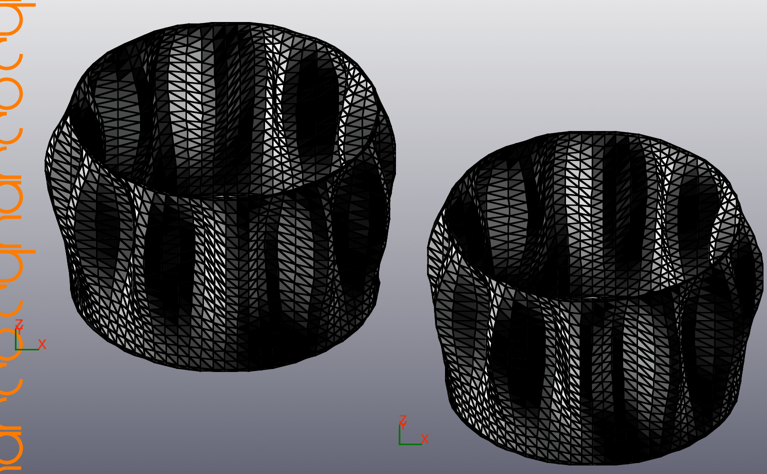
- Aluminum shell with geometrical imperfections (FE-model: stringer stiffened shell, 10.000 DOF)



Samples of random field

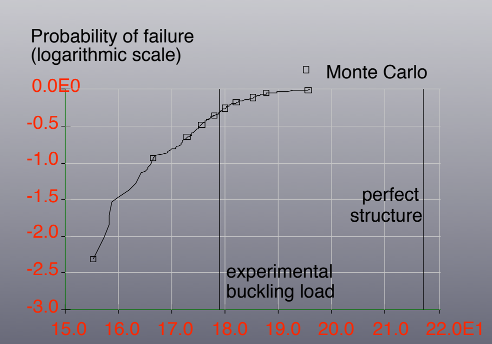


Buckling shapes



Results

- Probability of exceeding critical load as function of deterministic load factor



Robustness of a design

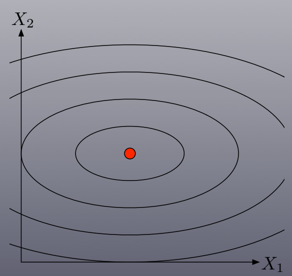
- Situation: random perturbation of
  - design parameters
  - other parameters
- Intuitively: The performance of a robust design is largely unaffected by random perturbations
- Statistical indicator: The coefficient of variation (COV) of the objective function and/or constraint values is smaller than the COV of the input variables

Optimization in the presence of randomness

- Optimization can provide e.g. mean values of the design variables
- Example objective function

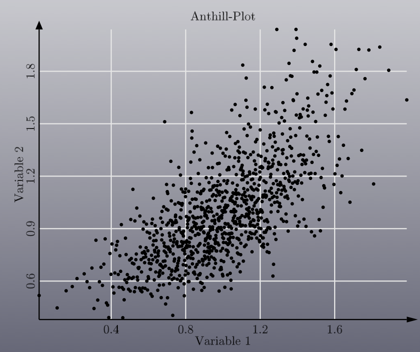
$$f(X_1, X_2) = 1 + (X_1 - 1)^2 + 2(X_2 - 1)^2$$

- The global minimum is located at (1,1)



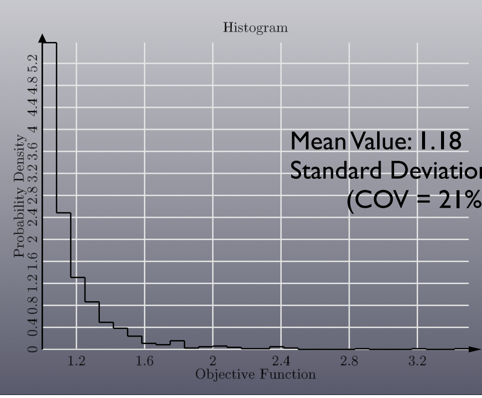
Case 1: Design variables have randomness

- Consider two correlated random variables (normal and log-normal with a COV = 30% and a correlation coefficient of 70%)



Histogram of objective function

- In this case, objective can only become larger than in the deterministic optimal case



Which variable is responsible?

- Correlation analysis
- X3 is the objective function
- Moderate correlation between input and output

ρ	X1	X2	X3
X1	1.00	0.70	0.14
X2	0.70	1.00	0.37
X3	0.14	0.37	1.00

Case 2: Design variables and objective function have randomness

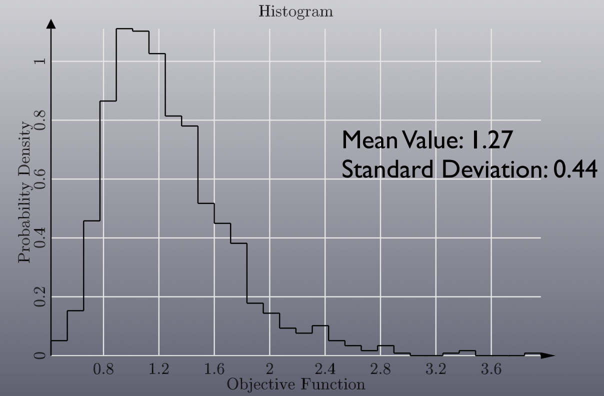
- Objective function similar as before

$$f(X_1, X_2, a, b) = a + (X_1 - b)^2 + 2(X_2 - 1)^2$$

- The global minimum is located at (b,1)
- Assume a and b to be random variables with mean value 1 and COV of 30%

Histogram of objective function value

- In this case, the objective may become smaller than in the deterministic case



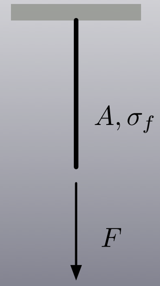
Correlation coefficients

- Significant correlation between input and output becomes visible

ρ	X1	X2	a	b	X3
X1	1.00	0.70	0.00	0.00	0.04
X2	0.70	1.00	0.00	0.00	0.16
a	0.00	0.00	1.00	0.00	0.68
b	0.00	0.00	0.00	1.00	0.20
X3	0.04	0.16	0.68	0.20	1.00

Safety oriented design problems

- Choose cross section and material strength in order to carry the load safely
- Consider uncertainties in all quantities
- "Reasonable" choice of level of safety
- Maintain economical design



Design optimization

- Cost function - include all factors relevant for the cost of a design

$$C_T = C_A(A) \cdot C_\sigma(\sigma_f) \rightarrow \text{Min.!}$$

- Safety Constraint - make sure that the design is sufficiently safe

$$A \cdot \sigma_f > F$$

- Utilization Constraint - make sure that the design can be effectively used

$$A < A_l$$

Deterministic Optimization

- As an example, let

$$C_T = A \cdot \sigma_f^2$$

- Unconstrained optimization: leads to the trivial solution

$$A = 0; \quad \sigma_f = 0$$

- Consider safety constraint, assume that solution lies at the boundary. This leads to

$$\sigma_f = 0; \quad A = \infty$$

- Including utilization constraint leads to

$$A = A_l; \quad \sigma_f = \frac{F}{A_l}$$

Reliability-based Optimization

- Expected value of cost as objective function

$$E[C_T] = E[C_A(A) \cdot C_\sigma(\sigma_f)] + C_f \cdot P_f + E[C_u(A)]$$

- Assume F and A to be Gaussian random variables with coefficients of variation 0.3 and 0.1, respectively
- Calculate probability of failure from

$$p_f = \Phi(-\beta); \quad \beta = \frac{\sigma_f \bar{A} - \bar{F}}{\sqrt{(0.1\sigma_f \bar{A})^2 + (0.3\bar{F})^2}}$$

Reliability-based Optimization (ctd.)

- Utilization cost is increasing with A, e.g.

$$C_u(A) = C_{util} \cdot A$$

- Final objective function (unconstrained problem!)

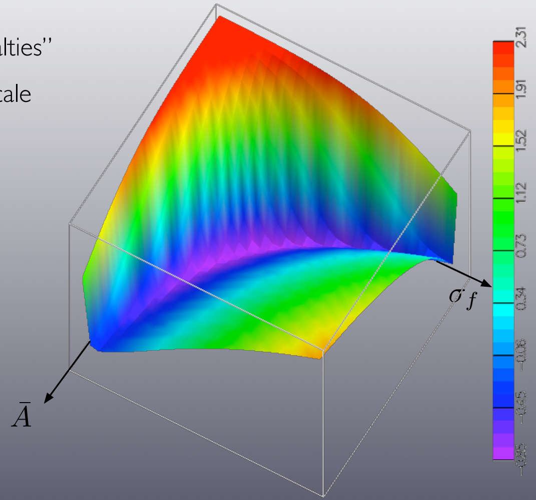
$$E[C_T] = \bar{A} \cdot \sigma_f^2 + C_f \cdot \Phi\left(\frac{\sigma_f \bar{A} - \bar{F}}{\sqrt{(0.1\sigma_f \bar{A})^2 + (0.3\bar{F})^2}}\right) + C_{util} \cdot \bar{A}$$

- Optimization parameters are

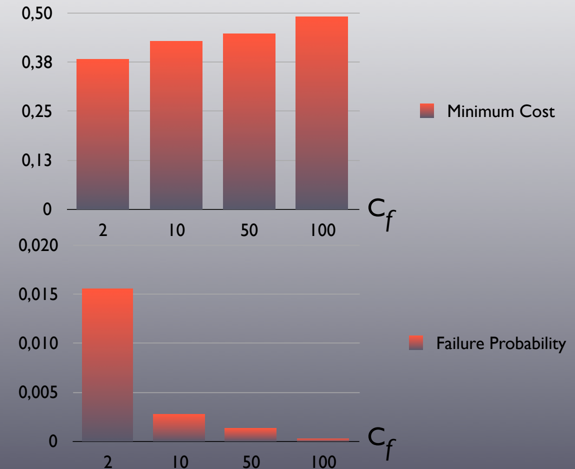
$$\bar{A}, \quad \sigma_f$$

Objective function

- Includes "penalties"
- Logarithmic scale

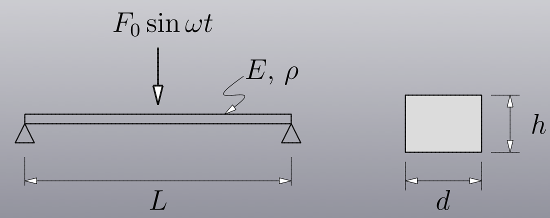


Effect of safety "penalty" (Cost of failure) on the optimal design



Optimization with Reliability Constraint

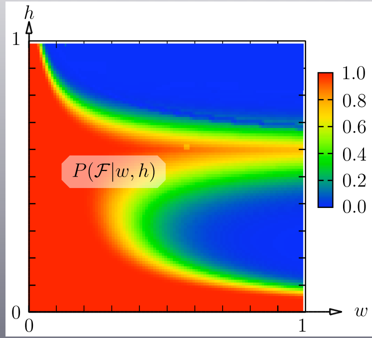
- Dynamic load characterized by random amplitude and frequency



- Minimize structural mass while keeping the probability of exceeding a dynamic deflection of 10 mm less than 1%.

Probability of Failure

- Compute probability of violating constraint by using FORM



- Disjoint admissible domains ➽ use Genetic algorithms

Response surface method

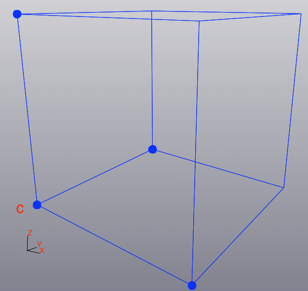
- Monte Carlo methods may become prohibitively expensive
- Limit state function may contain "noise" thus making FORM analysis difficult
- Prior knowledge about the general shape of the limit state function may be available
- Results from deterministic design variations should be re-utilized

Exploration of design space (Design of Experiments)

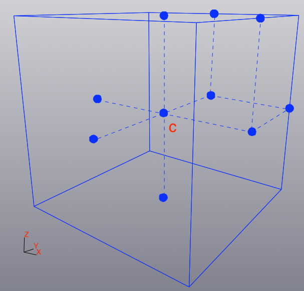
- Cover relevant regions in the space of random variables
- For reliability analysis, this is the region around the design point(s)
- Provide some redundancy to allow for error checking

Saturated designs

- Just as many support points as required



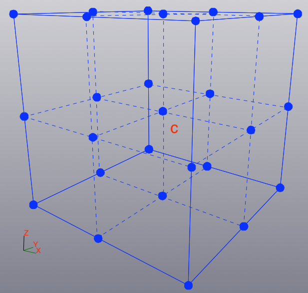
linear



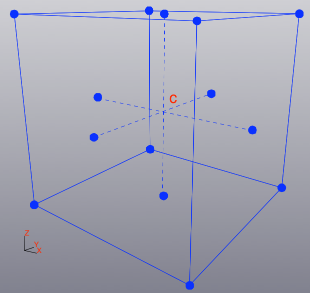
quadratic

Oversaturated designs

- More support points than minimally required



full factorial



central composite

Fitting by regression

- Response surface model

$$\eta = q(\theta_1, \theta_2, \dots, \theta_p; x_1, x_2, \dots, x_n) = q(\boldsymbol{\theta}; \mathbf{x})$$

- Points of experiments

$$\mathbf{x}^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})', \qquad k = 1, 2, \dots, m$$

- Measured responses

$$z^{(k)}; \quad k = 1, 2 \dots, m$$

- Regression

$$s(\boldsymbol{\theta}) = \sum_{k=1}^{m} \left(z^{(k)} - q(\boldsymbol{\theta}; \mathbf{x}^{(k)}) \right)^2 \rightarrow \text{Min.!}$$

Linear models

- Dependence on free parameters is linear, dependence on design variables is generally non-linear

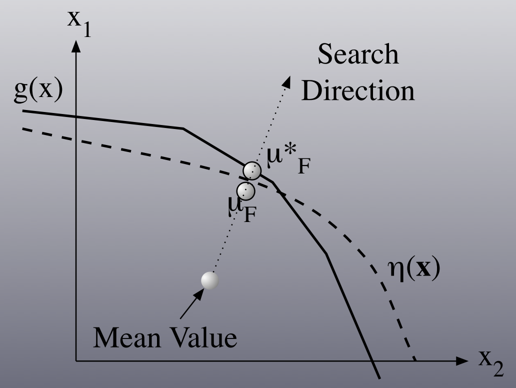
$$\eta = \theta_1 q_1(\mathbf{x}) + \theta_2 q_2(\mathbf{x}) + \dots + \theta_p q_p(\mathbf{x})$$

$$\mathbf{z} = \begin{bmatrix} z^{(1)} \\ z^{(2)} \\ \vdots \\ z^{(m)} \end{bmatrix} = \begin{bmatrix} q_1(\mathbf{x}^{(1)}) & q_2(\mathbf{x}^{(1)}) & \dots & q_p(\mathbf{x}^{(1)}) \\ q_1(\mathbf{x}^{(2)}) & q_2(\mathbf{x}^{(2)}) & \dots & q_p(\mathbf{x}^{(2)}) \\ \vdots & \vdots & & \vdots \\ q_1(\mathbf{x}^{(m)}) & q_2(\mathbf{x}^{(m)}) & \dots & q_p(\mathbf{x}^{(m)}) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{bmatrix} + \begin{bmatrix} \varepsilon^{(1)} \\ \varepsilon^{(2)} \\ \vdots \\ \varepsilon^{(m)} \end{bmatrix} = \mathbf{Q}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$$

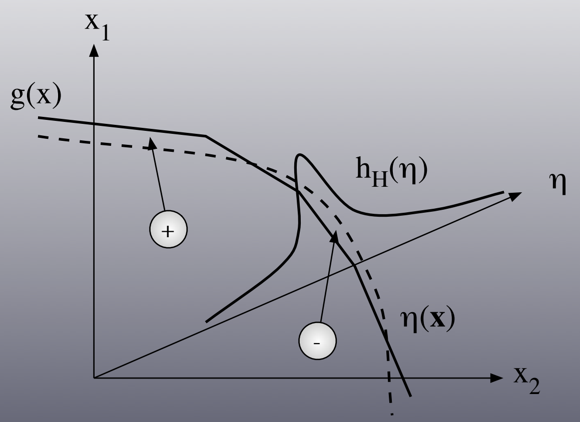
Improvement of accuracy

- Adaptation of response surface
- Pointwise checks of accuracy
- Regional checks of accuracy

Check accuracy of approximate design point

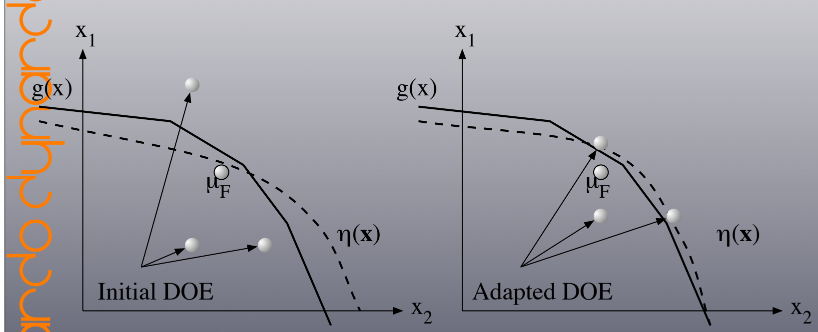


Correction by sampling near the approximate limit state



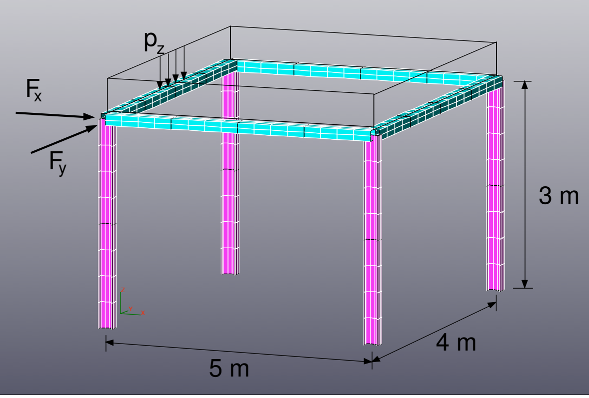
Adaptation of response surface

- Requires adaptation of the DOE scheme
- Shift and Shrink



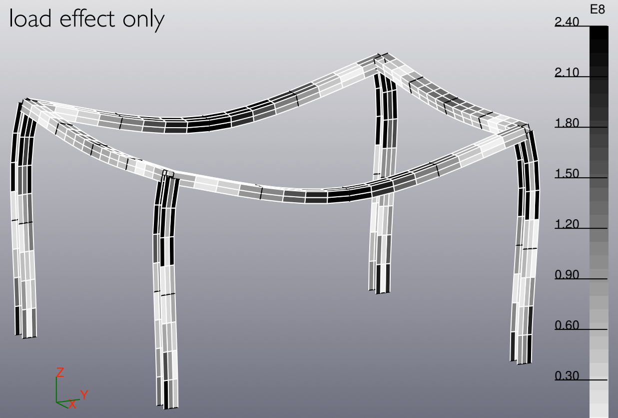
Example: nonlinear frame structure

- Random loads, deterministic structure
- Linear-elastic ideally-plastic material



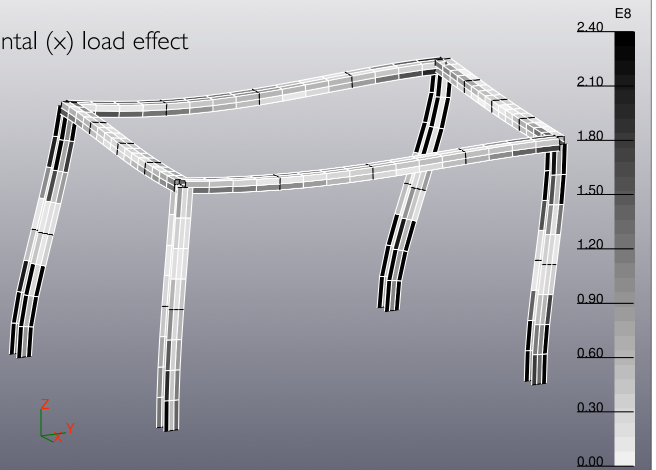
Stress distribution at limit state

- Vertical load effect only



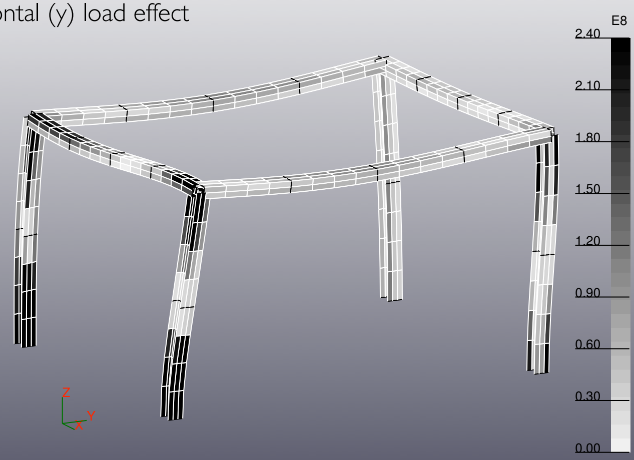
Stress distribution at limit state

- Horizontal (x) load effect



Stress distribution at limit state

- Horizontal (y) load effect



Distribution of random variables

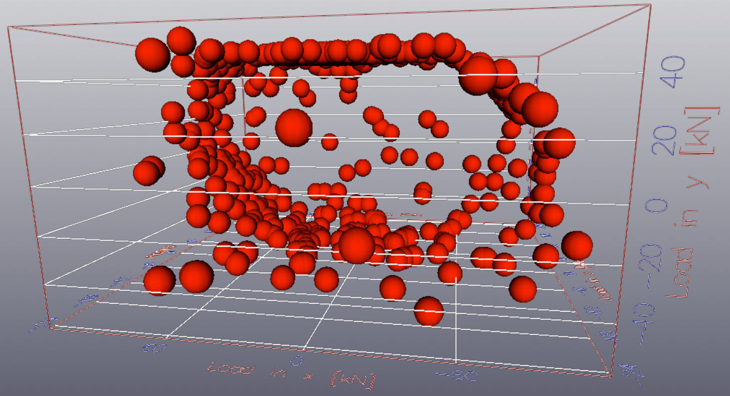
Number	Variable	Mean	Standard Deviation	Type
1	p_z [kN/m]	12.0	0.8	Gumbel
2	F_x [kN]	30.0	2.4	Gumbel
3	F_y [kN]	40.0	3.2	Gumbel

- Exact failure probability (directional sampling)

$$P(F) = 2.3 \cdot 10^{-4}$$

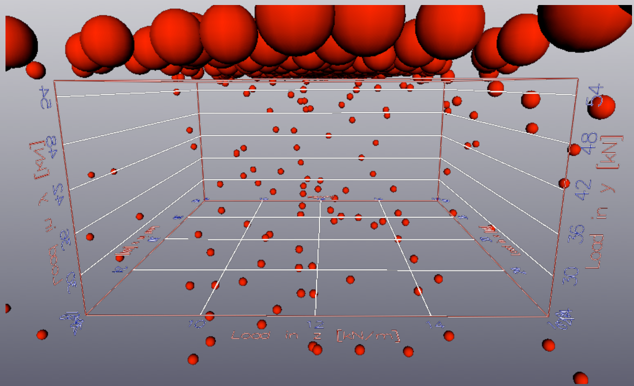
Visualization of limit state function

- Ranges ±5 standard deviations around the mean



Visualization of limit state function

- Details near the design point



Support points for quadratic response surface

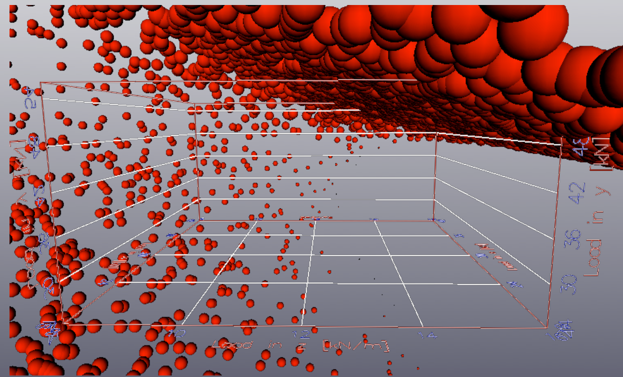
Number	p_z [kN/m]	F_x [kN]	F_y [kN]	g
1	12.000	30.000	40.000	1
2	21.513	30.000	40.000	0
3	-21.516	30.000	40.000	0
4	12.000	113.180	40.000	0
5	12.000	-81.094	40.000	0
6	12.000	30.000	59.082	0
7	12.000	30.000	-59.087	0
8	19.979	109.790	40.000	0
9	13.527	30.000	59.084	0
10	12.000	45.275	59.094	0

- Approximate failure probability

$$P(F) = 1.7 \cdot 10^{-4}$$

Visualization of response surface

- Details near the design point



Concluding remarks

- Combination of optimization and stochastic analysis opens new perspectives
- Robustness analysis helps to detect "weak spots" in a design
- Reliability analysis provides a rational basis for safety-related design decisions
- Stochastic analysis requires substantial computer power
- Response surfaces require careful checking procedures

Thank you very much
for your kind attention!