Basis concepts of robust design optimization

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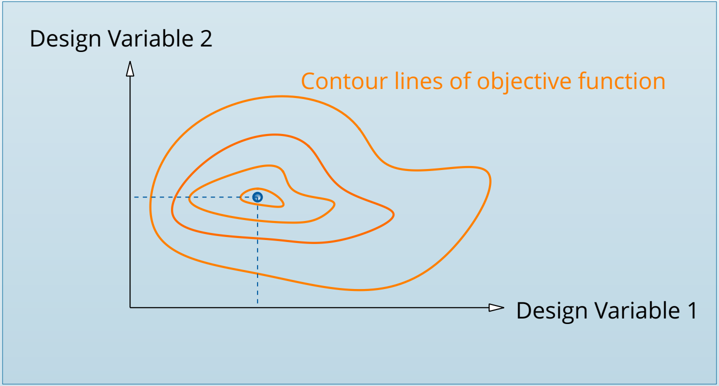
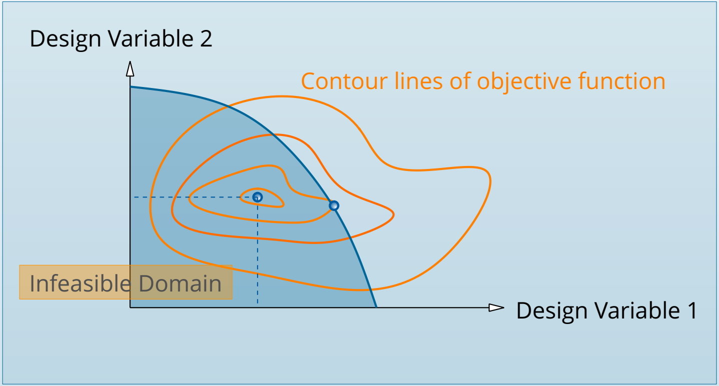
Overview

- Introduction
- Uncertainties in Optimization
- Motivating Example
- Probability theory
- Estimation
- Failure probability
- Sampling methods
- Response Surface Method
- Quality of Meta-models
- Example

Uncertainties in optimization

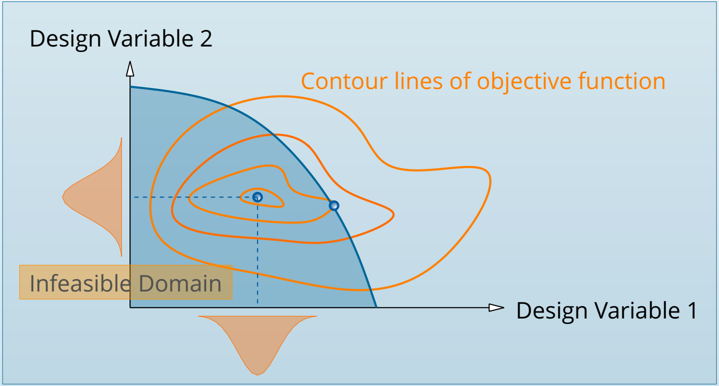


Uncertainties in optimization



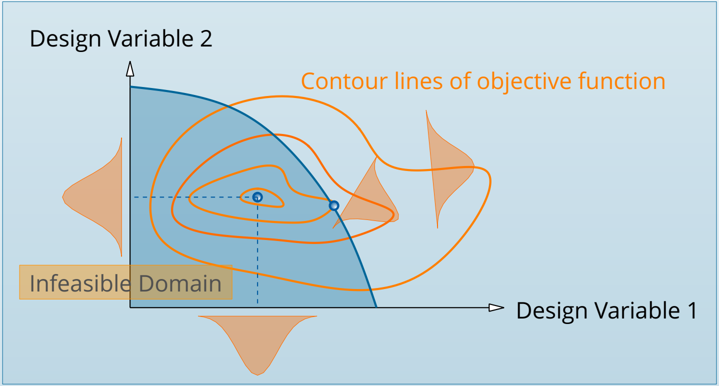
Uncertainties in optimization

- Design variables (e.g. manufacturing tolerances)



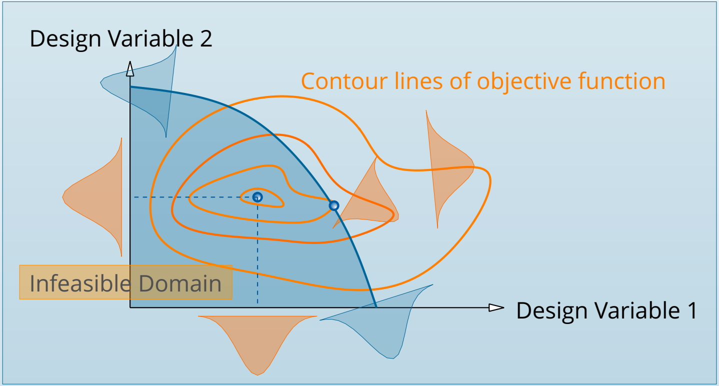
Uncertainties in optimization

- Design variables (e.g. manufacturing tolerances)
- Objective function (e.g. tolerances, external factors)



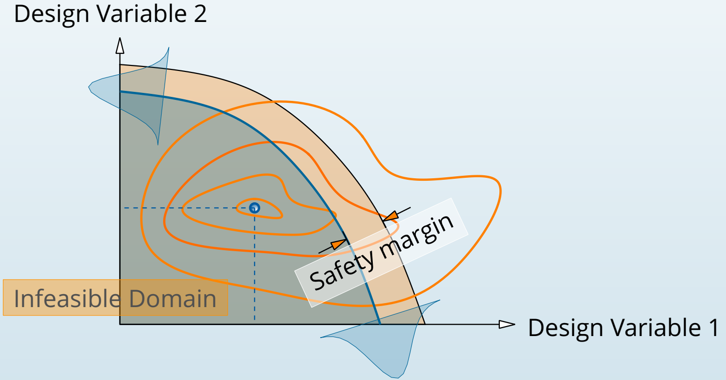
Uncertainties in optimization

- Design variables (e.g. manufacturing tolerances)
- Objective function (e.g. tolerances, external factors)
- Constraints (e.g. tolerances, external factors)

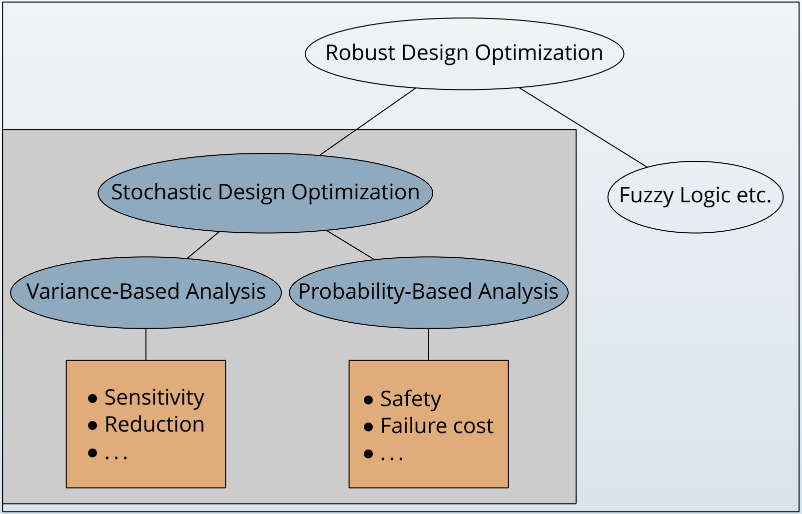


Traditional design approach

- Introduce "safety factors" into the constraints
- Leads to results satisfying safety requirement, but not necessarily optimal designs

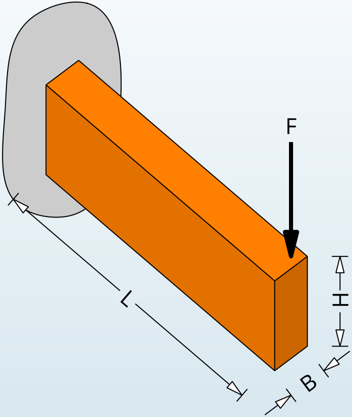


Tools for optimal robust design



Motivating Example 1

- Deterministic problem



- Minimize cross section area of a cantilever $A = B \cdot H$
- Constraint 1: limited vertical deflection w

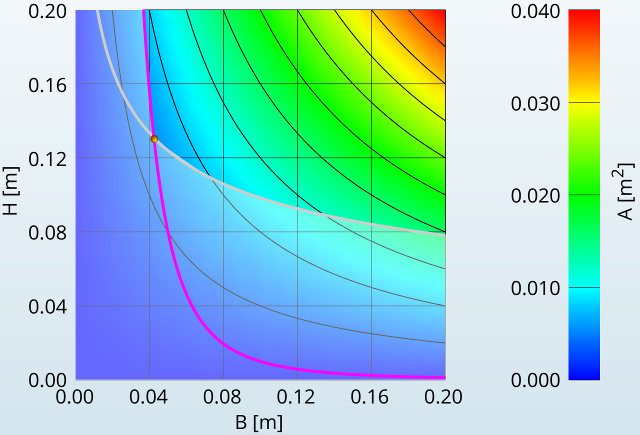
$$w = \frac{4FL^3}{EBH^3} \leq w_0$$

- Constraint 2: Sufficient stability in lateral torsional buckling

$$F_{cr} = 0.4741 \frac{EB^3H}{L^2} \geq \lambda F$$

Motivating Example 2

- Numerical solution for
  $L = 1$ m, $E = 210$ MPa, $F = 500$ N, $\lambda = 2$, $w_0 = 0.1$ m:
  Solution: $B = 0.0425$ m, $H = 0.1309$ m, $A = 0.00556$ m^2



Motivating Example 3

- Stochastic problem
  - F is a random variables with mean value $\bar{F}$ and standard deviation σ_F (assumed normally distributed)
  - Satisfy constraint conditions with certain probability $P_i \approx 1$

$$\mathbf{P}[C_1 \leq 0] = \mathbf{P}\left[\frac{4FL^3}{EBH^3} - w_0 \leq 0\right] = \mathbf{P}\left[F \leq \frac{EBH^3 w_0}{4L^3}\right] \geq P_1$$

$$\mathbf{P}[C_2 \leq 0] = \mathbf{P}\left[\lambda F - 0.4741\frac{EB^3H}{L^2} \leq 0\right] = \mathbf{P}\left[F \leq \frac{0.4741EB^3H}{\lambda L^2}\right] \geq P_2$$

  - B, H, L may be random, too
  - Objective function

$$A = BH \to \text{Min.!}$$

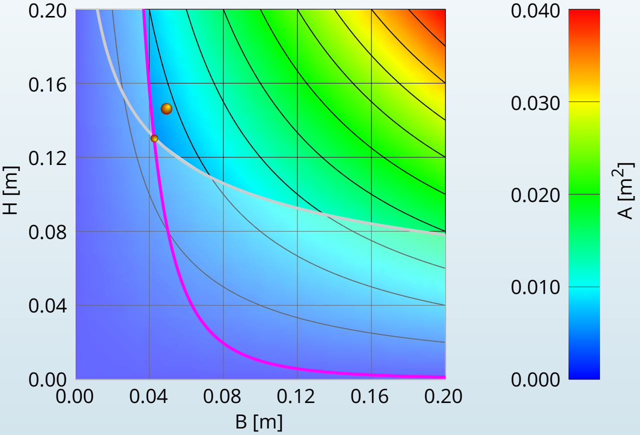
  - Constraint conditions

$$\mathbf{P}[C_1 \geq 0] \leq 1 - P_1 = P_{F_1}$$

$$\mathbf{P}[C_2 \geq 0] \leq 1 - P_2 = P_{F_2}$$

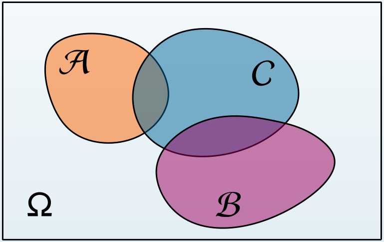
Motivating Example 4

- Numerical solution for $\bar{F} = 500$ N, $\sigma_F = 100$ N, $P_{F_1} = 10^{-3}$, $P_{F_2} = 10^{-4}$
- B = 0.0493 m, H = 0.1462 m, A = 0.00721 m^2 (30% more)



Probability

- Events

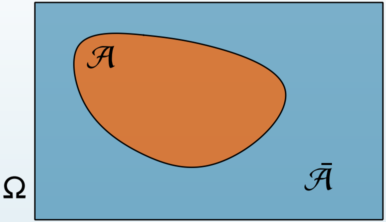


- Axioms(Kolmogorov)

$$
\begin{aligned}
\text{I}: &\quad 0 \le \mathrm{P}[\mathcal{A}] \le 1 \\
\text{II}: &\quad \mathbf{P}[\Omega] = 1 \\
\text{III}: &\quad \mathbf{P}[\mathcal{A} \cup \mathcal{B}] = \mathbf{P}[\mathcal{A}] + \mathbf{P}[\mathcal{B}]
\end{aligned}
$$

$$\mathbf{P}[\mathcal{A} \cup \mathcal{C}] = \mathbf{P}[\mathcal{A}] + \mathbf{P}[\mathcal{C}] - \mathbf{P}[\mathcal{A} \cap \mathcal{C}]$$

Complementary Event



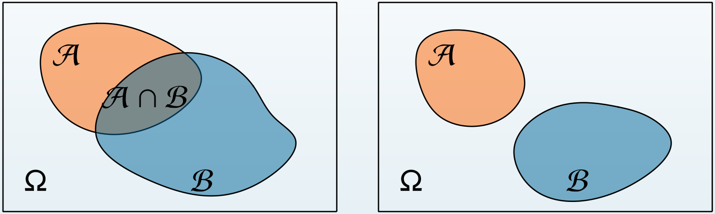
- An event can either happen or not happen

$$\mathbf{P}[\mathcal{A}] + \mathbf{P}[\bar{\mathcal{A}}] = \mathbf{P}[\Omega] = 1$$

- An event cannot happen and not happen at the same time

$$\mathbf{P}[\mathcal{A} \cap \bar{\mathcal{A}}] = \mathbf{P}[\emptyset] = 0$$

Conditional Probability



- Definition

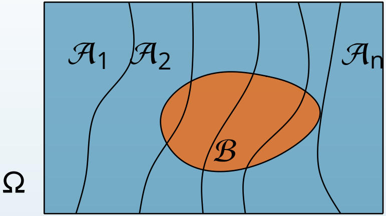
$$\mathbf{P}[\mathcal{A}|\mathcal{B}] = \frac{\mathbf{P}[\mathcal{A} \cap \mathcal{B}]}{\mathbf{P}[\mathcal{B}]}$$

- Independence

$$\mathbf{P}[\mathcal{A}|\mathcal{B}] = \mathbf{P}[\mathcal{A}]$$

$$\rightarrow \mathbf{P}[\mathcal{A} \cap \mathcal{B}] = \mathbf{P}[\mathcal{A}]\mathbf{P}[\mathcal{B}]$$

Decomposition of event space



- Total probability

$$\mathbf{P}[\mathcal{B}] = \mathbf{P}[\mathcal{B}|\mathcal{A}_1]\,\mathrm{P}[\mathcal{A}_1] + \ldots + \mathbf{P}[\mathcal{B}|\mathcal{A}_n]\mathbf{P}[\mathcal{A}_n]$$

- Bayes' theorem

$$\mathbf{P}[\mathcal{A}_i|\mathcal{B}] = \frac{\mathbf{P}[\mathcal{B}|\mathcal{A}_i]\mathbf{P}[\mathcal{A}_i]}{\mathbf{P}[\mathcal{B}|\mathcal{A}_1]\mathbf{P}[\mathcal{A}_1] + \ldots + \mathbf{P}[\mathcal{B}|\mathcal{A}_n]\mathbf{P}[\mathcal{A}_n]}$$

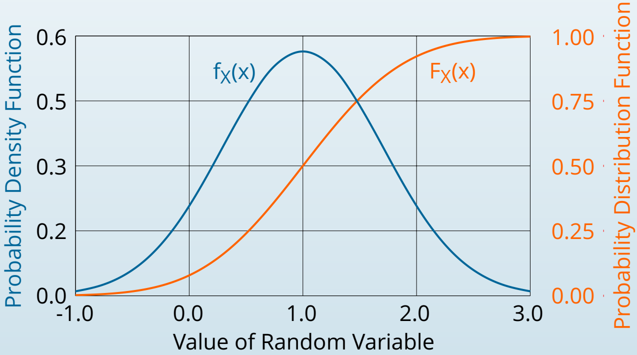
Random Variables

- Distribution function

$$F_X(x) = \mathbf{P}[X < x]; \quad \lim_{x\to-\infty} F_X(x) = 0; \quad \lim_{x\to+-\infty} F_X(x) = 1$$

- Probability density function

$$f_X(x) = \frac{d}{dx} F_X(x)$$



Expected values

- Definition

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- Mean value

$$\bar{X} = \mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Variance (square of standard deviation)

$$\sigma_X^2 = \mathbf{E}[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx$$

- Coefficient of variation (dimensionless)

$$V_X = \frac{\sigma_X}{\bar{X}}; \quad \bar{X} \neq 0$$

- Expectation is a linear operator

$$\mathbf{E}[g + h] = \mathbf{E}[g] + \mathbf{E}[h]; \quad \mathbf{E}[\lambda g] = \lambda \mathbf{E}[g]$$

Estimation

- Estimator Γ for an unknown parameter γ (e.g. mean value) from independent observations $X_k; k = 1 \dots n$
- Consistency

$$\Gamma : \Gamma_n = \Gamma(X_1, \dots X_n)$$

$$\forall \epsilon > 0 : \quad \lim_{n\to\infty} P[|\Gamma_n - \gamma| < \epsilon] = 1$$

- Unbiasedness

$$\mathbf{E}[\Gamma_n] = \gamma$$

- Asymptotic unbiasedness

$$\lim_{n\to\infty} \mathbf{E}[\Gamma_n] = \gamma$$

- Any estimate based on finite sample size contains some uncertainty which should be made sufficiently small (usually by adjusting the sample size)

Estimation error

- Limited number of samples leads to random deviation of the estimate from the true expected value
- Example: estimator for the mean value

$$m_X = \frac{1}{n} \sum_{i=1}^{n} X_i$$

- Variance of the estimated value

$$\sigma_m^2 = \mathbf{E}[(m - \bar{X})^2]$$

- Estimator for the variance of the mean value estimator

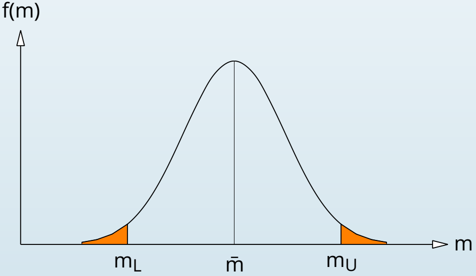
$$S_m^2 = \frac{1}{n(n-1)} \sum_{i=1}^{n} (m - X_i)^2 = \frac{1}{n} S_X^2$$

Confidence interval

- Statistical error (standard deviation) of the estimator

$$s_m = \frac{s_X}{\sqrt{n}}$$

- Assume normally distributed error → Compute confidence interval for estimated value



$$\mathbf{P}[m_L \leq m \leq m_U] = 1 - \alpha$$

Two distribution functions

- Normal (Gaussian) distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right]; \quad -\infty < x < \infty$$

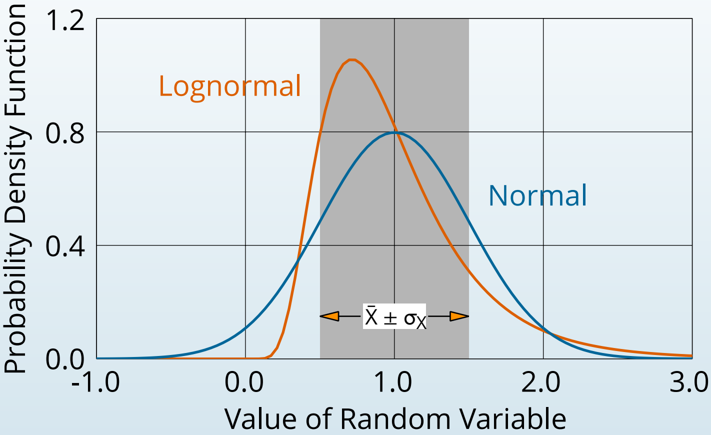
$$F_X(x) = \Phi\left(\frac{x-\bar{X}}{\sigma_X}\right); \quad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{u^2}{2}\right) du$$

- Log-normal distribution

$$f_X(x) = \frac{1}{x\sqrt{2\pi}s} \exp\left[-\frac{(\log \frac{x}{\mu})^2)}{2s^2}\right]; \quad 0 \le x < \infty$$

$$F_X(x) = \Phi\left(\frac{\log \frac{x}{\mu}}{s}\right); \quad \mu = \bar{X} \exp\left(-\frac{s^2}{2}\right); \quad s = \sqrt{\ln\left(\frac{\sigma_X^2}{\bar{X}^2} + 1\right)}$$

Normal and log-normal density functions



Random Vectors

- Collect all random variables into a random vector

$$\mathbf{X} = [X_1, X_2, \dots X_n]^T$$

- Mean value by applying expectation operator to all components

$$\bar{\mathbf{X}} = \mathbf{E}[\mathbf{X}] = [\bar{X}_1, \bar{X}_2, \dots \bar{X}_n]^T$$

- Covariance matrix

$$\mathbf{C_{XX}} = \mathbf{E}[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^T]$$

- Coefficient of correlation

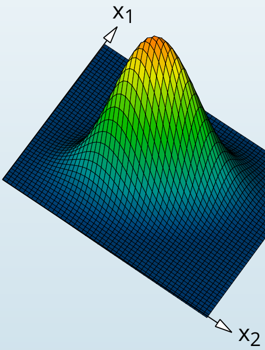
$$\rho_{ik} = \frac{\mathbf{E}[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]}{\sigma_{X_i}\sigma_{X_k}}$$

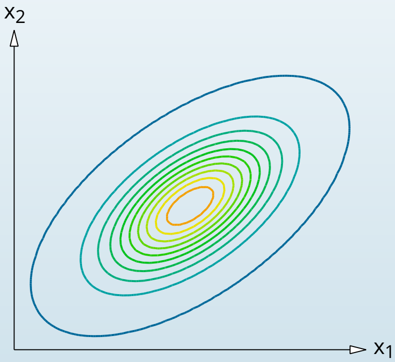
Joint probability density function

- Multi-dimensional normal distribution

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{C}_{\mathbf{XX}}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\bar{\mathbf{X}})^T \mathbf{C}_{\mathbf{XX}}^{-1}(\mathbf{x}-\bar{\mathbf{X}})\right]; \ \mathbf{x} \in \mathbb{R}^n$$

- Two-dimensional case





Nataf model

- Transformation of correlated non-Gaussian random variables (ρ_{ik}) to correlated standard Gaussian variables (ρ'_{ik})

$$\{X_i; f_{X_i}(x_i)\} \leftrightarrow \{V_i; \varphi(v_i)\}$$

- Mapping

$$V_i = \Phi^{-1}[F_{X_i}(X_i)]$$

- Properties

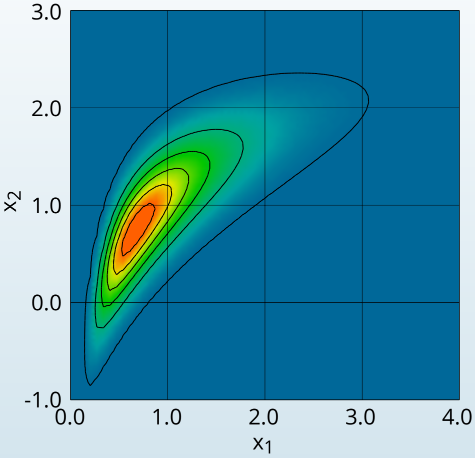
$$\mathbf{E}[V_i] = 0; \quad \mathbf{E}[V_i^2] = 1; \quad \mathbf{E}[V_i V_k] = \rho'_{ik}$$

- Assumption of a multi-dimensional Gaussian distribution

$$f_{\mathbf{V}}(\mathbf{v}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det \mathbf{R_{VV}}}} \exp\left(-\frac{1}{2}\mathbf{v}^T \mathbf{R}_{\mathbf{VV}}^{-1} \mathbf{v}\right)$$

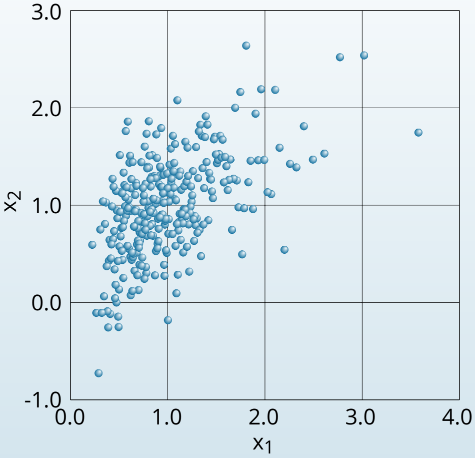
Joint probability density function

- Example: two correlated random variables
- X_1 . . . Lognormally distributed
- X_2 . . . Normally distributes
- Both variables have mean values 1, standard deviations 0.5, correlation $\rho_{12} = 0.5$



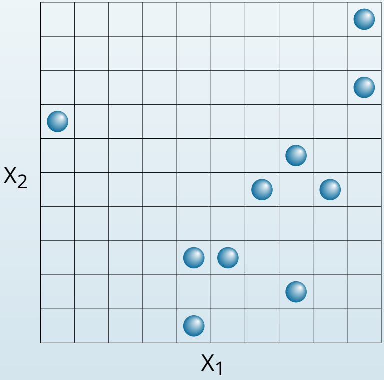
Simulation of samples

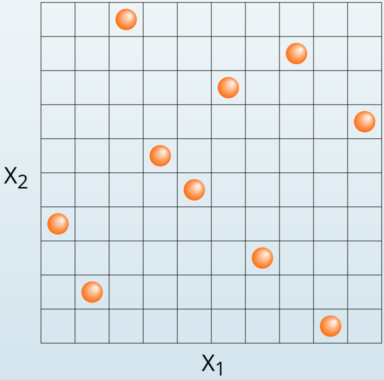
- Example: two correlated random variables as before
- Correlation $\rho_{12} = 0.5$
- 1000 samples



Plain Monte Carlo vs. Latin Hypercube

- Special considerations required for small sample size
- 10 samples of uniformly distributed independent random variables
- Quasi-random sampling provides better coverage of space



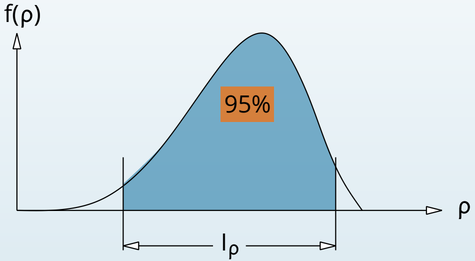


PMC

LHS

Estimation of correlations

- Repeated simulations lead to different results → estimator for ρ is randomly distributed (but not normal)

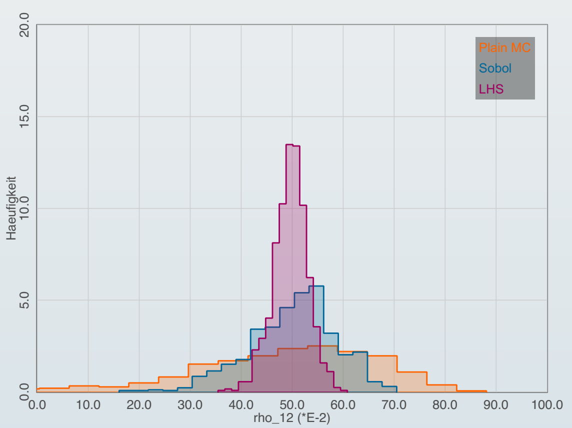


$$I_\rho = [\tanh(z_{ij} - \frac{z_c}{\sqrt{N-3}}),\ \tanh(z_{ij} + \frac{z_c}{\sqrt{N-3}})]$$

$$z_{ij} = \frac{1}{2}\log\frac{1+\rho_{ij}}{1-\rho_{ij}}; \quad z_c = \Phi^{-1}(1-\alpha'/2)$$

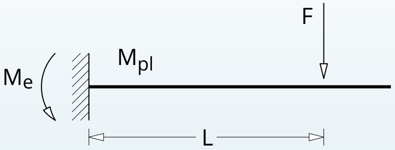
Example

- Repeated simulation of two correlated Gaussian variables
- Estimate coefficient of correlation from samples
- Perform statistics on the estimators



Reliability analysis

- Mechanical system



- Failure condition

$$\mathcal{F} = \{(F, L, M_{pl}) : FL \geq M_{pl}\} = \{(F, L, M_{pl}) : 1 - \frac{FL}{M_{pl}} \leq 0\}$$

- Failure probability

$$\mathbf{P}(\mathcal{F}) = \mathbf{P}[\{\mathbf{X} : g(\mathbf{X}) \leq 0\}$$

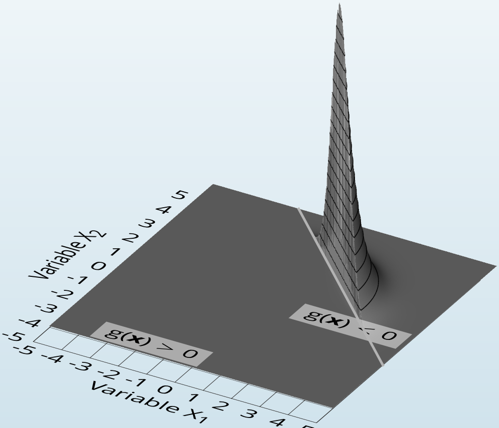
$$\mathbf{P}(\mathcal{F}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} I_g(x) f_{X_1 \dots X_n} dx_1 \dots dx_n$$

$I_g(x_1 \dots x_n) = 1$ if $g(x_1 \dots x_n) \leq 0$ and $I_g(.) = 0$ else

Computational Challenge

- Integrand is non-zero only in a small region
- Difficult to find appropriate integration points
- Example in standard Gaussian space

$$g(x_1, x_2) = 3 - x_1 - x_2$$



First order reliability method (FORM)

- Transformation to standard Gaussian space (here: Rosenblatt transform for Nataf-model)

$$Y_i = \Phi^{-1}[F_{X_i}(X_i)]; \quad i = 1 \dots n$$

$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{Y}; \quad \mathbf{C_{YY}} = \mathbf{L}\mathbf{L}^T$$

- Inverse transformation

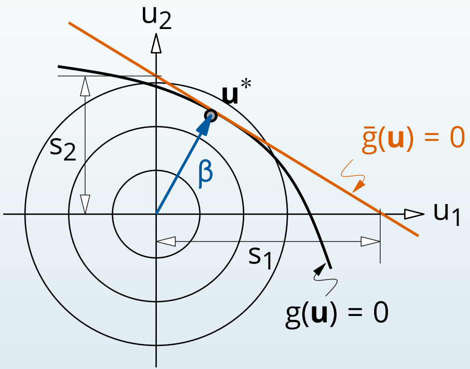
$$X_i = F_{X_i}^{-1}\left[\Phi\left(\sum_{k=1}^{n} L_{ik} U_k\right)\right]$$

- Computation of "design point"

$$\mathbf{u}^* : \mathbf{u}^T\mathbf{u} \to \text{Min.}; \quad \text{subject to} : g[\mathbf{x}(\mathbf{u})] = 0$$

- Linearize the limit state function at the design point in standard Gaussian space

FORM Procedure



$$\bar{g} : -\sum_{i=1}^{n} \frac{u_i}{s_i} + 1 = 0; \qquad \sum_{i=1}^{n} \frac{1}{s_i^2} = \frac{1}{\beta^2}$$

$$\mathbf{P}(\mathcal{F}) = \Phi(-\beta)$$

Monte Carlo estimation

- Write failure probability as expectation

$$\mathbf{P}(\mathcal{F}) = p_f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} I_g(x) f_{X_1 \dots X_n} dx_1 \dots dx_n$$

- Indicator function

$$I_g(x_1 \dots x_n) = \begin{cases} 1 \text{ for } g(x_1 \dots x_n) \leq 0 \\ 0 \text{ else} \end{cases}$$

- Consistent and unbiased estimator (arithmetic mean)

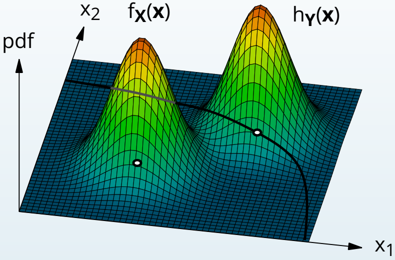
$$\bar{p}_f = \frac{1}{m} \sum_{k=1}^{m} I_g(\mathbf{x}^{(k)})$$

- Variance of estimator

$$\sigma^2_{\bar{p}_f} = \frac{p_f}{m} - \frac{p_f^2}{m} \approx \frac{p_f}{m} \rightarrow \sigma_{\bar{p}_f} = \sqrt{\frac{p_f}{m}}$$

Importance sampling

- Simulation density



- Estimator of failure probability

$$\bar{P}(\mathcal{F}) = \frac{1}{m} \sum_{k=1}^{m} \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} I_g(\mathbf{x}) = \mathbf{E}\left[\frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} I_g(\mathbf{x})\right]$$

- Variance of estimator

$$\sigma^2_{\bar{P}(\mathcal{F})} = \frac{1}{m} \mathbf{E}\left[\frac{f_{\mathbf{X}}(\mathbf{x})^2}{h_{\mathbf{Y}}(\mathbf{x})^2} I_g(\mathbf{x})\right]$$

Importance sampling at the design point

- Determine design point $\mathbf{u}^*$ in standard Gaussian space (e.g. using FORM)
- Construct a multi-dimensional standard Gaussian sampling density centered at the design point with unit covariance matrix (identical to that of the actual random variables in standard Gaussian space)

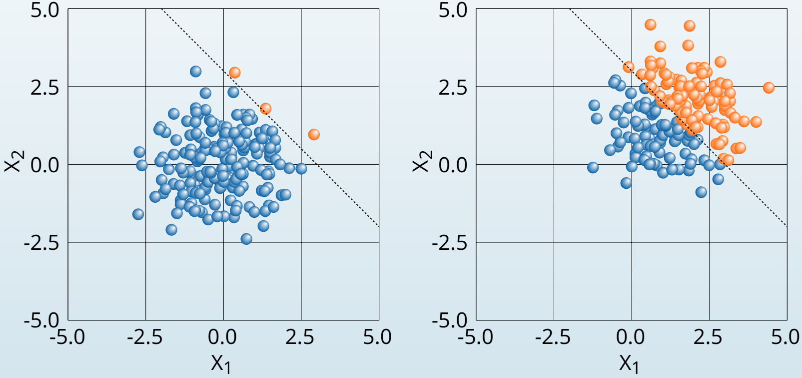
$$h_{\mathbf{Y}}(\mathbf{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2}(\mathbf{u} - \mathbf{u}^*)^T(\mathbf{u} - \mathbf{u}^*)\right]$$

- Carry out random sampling and estimation of the failure probability

Example

- Two standard Gaussian random variables

$$g(X_1, X_2) = 3 - X_1 - X_2; \quad \mathbf{x}^* = [1.5, 1.5]^T$$



Monte Carlo

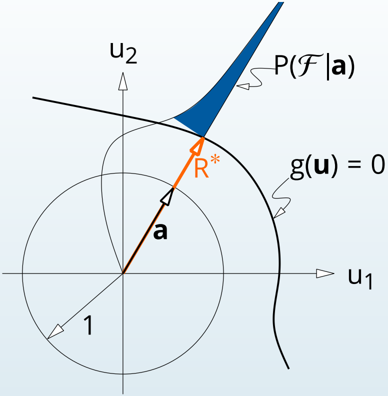
Importance sampling

Directional sampling

- Transformation into standard Gaussian space
- Generate random unit direction vector
- Compute the distance from the origin to the failure domain in this direction (typically using bisection)
- Compute conditional failure probability for this direction (Chi-Square distribution)
- Statistical analysis (estimation of mean and variance)

Procedure

- Conditional failure probability $P(\mathcal{F}|\mathbf{a})$ for one direction $\mathbf{a}$

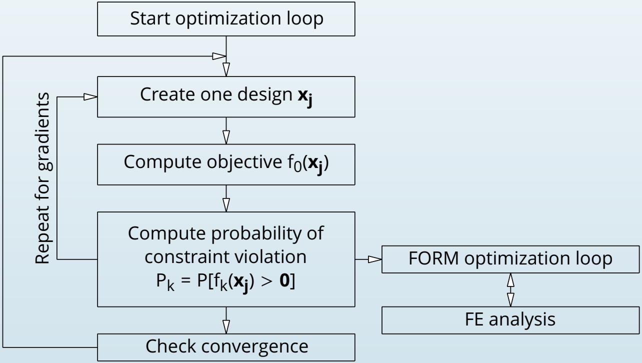


$$P(\mathcal{F}|\mathbf{a}) = \int_{R^*(\mathbf{a})}^{\infty} f_{R|\mathbf{A}}(r|\mathbf{a})dr =$$

$$= S_n r^{n-1} \frac{1}{\pi^{\frac{n}{2}}} \exp\left(-\frac{r^2}{2}\right) dr = 1 - \chi^2_n[R^*(\mathbf{a})^2]$$

Robust Optimization Procedure

- Outer optimization loop controls the structural design
- Probability of constraint violation computed by FORM
- Inner optimization driven by random variables
- Both loops need gradients . . .



The need for speed ...

- Complex system (many parameters, computationally expensive, slow, ...)
- Needed: Fast and reasonably accurate response prediction (e.g. for real-time applications such as control systems)
- Possible choices:
  - Reduce model complexity based on essential physical features (“reduced order model”)
  - Replace model based on mathematical simplicity (“metamodel”)
- Stochastic analysis needs to be very efficient

Reduced order model

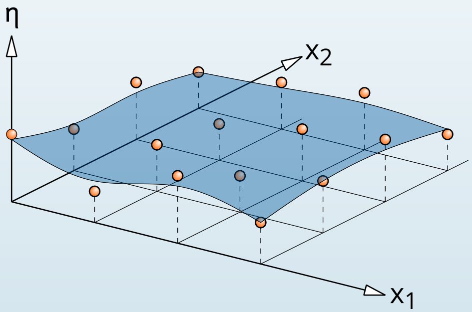
- Need to understand and represent physics
- May be applicable for many different load cases
- Very suitable for time dependent phenomena (structural dynamics, convection-diffusion processes)
- Can be difficult in the presence of strong nonlinearity
- Typical examples
  - Modal analysis
  - Proper orthogonal decomposition (POD)

Metamodel

- Mathematically formulated black box
- Suitable for arbitrarily nonlinear input-output relations
- Requires extensive training data
- Very difficult to extrapolate
- Time-dependent problems may be tricky
- Typical example: Linear or quadratic response surface model

Response surface method

- Reduce computational effort by replacing expensive FE analyses
- Establish meta-models in terms of simple mathematical functions
- Fit model parameters to FE solution using regression analysis



Regression models 1

- Mathematical formulation for response surfaces is closely related to linear regression and interpolation modeling
- Response surface model is based on linear regression if its functional form if linear in the unknown parameters p_k, i.e.

$$\eta(\mathbf{x}) = \sum_{k=1}^{n} p_k f_k(\mathbf{x})$$

- Sequence of input values $\mathbf{x}_i$, $i = 1 \ldots m$ and corresponding model output values y_i, $i = 1 \ldots m$
- Determine parameters p_k can be determined by solving the least squares problem

$$S^2 = \sum_{i=1}^{m} \left[y_i - \eta(\mathbf{x}_i)\right]^2 \to \text{Min.!}$$

Regression models 2

- Together with the linear regression model this results in

$$S^2 = \sum_{i=1}^{m} \left[y_i - \sum_{k=1}^{n} p_k f_k(\mathbf{x}_i) \right]^2 \rightarrow \text{Min.!}$$

- If the number of parameters n is equal to the number of data pairs m, then the regression model becomes an interpolation model.
- Global functions are functions not localizing in certain areas (such as polynomials)
- Linear polynomial function

$$\eta_l(\mathbf{x}) = p_0 + \sum_{k=1}^{n} p_k x_k$$

- Quadratic model

$$\eta_q(\mathbf{x}) = p_0 + \sum_{k=1}^{n} p_k x_k + \sum_{k=1}^{n} \sum_{j=1}^{n} p_{kj} x_k x_j$$

Regression models 3

- Localized models such as radial basis functions

$$\eta_r(\mathbf{x}) = \sum_{k=1}^{n} p_k \phi_k(\mathbf{x}, \mathbf{x}_k)$$

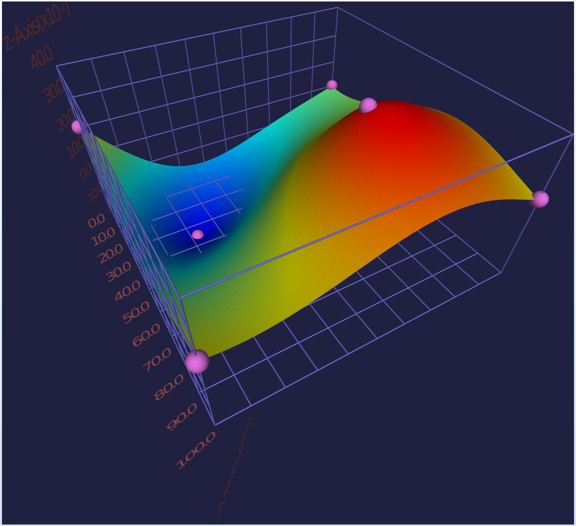
in which $\phi_k(\mathbf{y}) = \phi_k(\|\mathbf{y}\|) = \phi_k(r)$ are functions depending only on the magnitude of the vector argument and $\mathbf{x}_k$ are the localization points of the RBF functions. If the localization points coincide with the data points, then this model is interpolating. Otherwise it is a linear regression model.

- Thin plate splines

$$\phi(r) = r^2 \log r$$

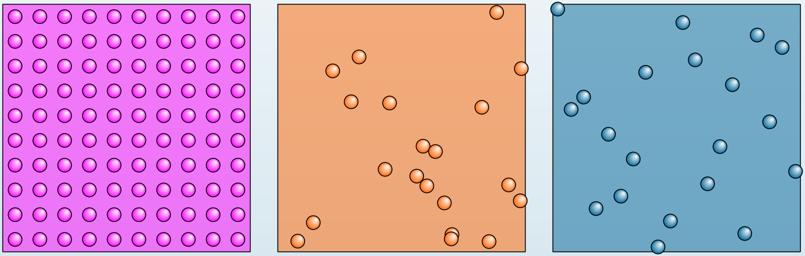
Regression models 4

- Example (6 data points in 2D)



Design of Experiments (DoE)

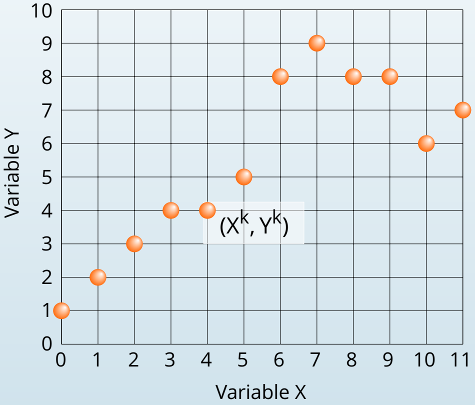
- Explore range of variables by numerical experiments
- Cover range of all variables as uniformly as possible
- Keep number of experiments small



Factorial design   Monte Carlo Sampling   Latin Hypercube Sampling

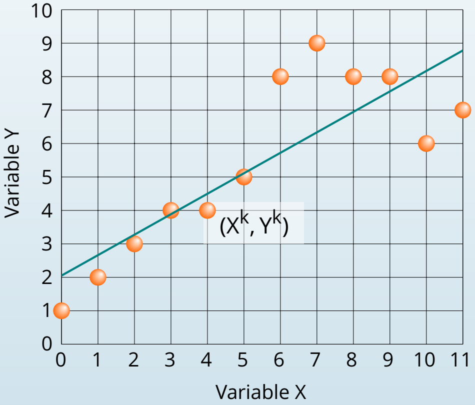
Example

- Adapt 1D metamodel to 12 data points



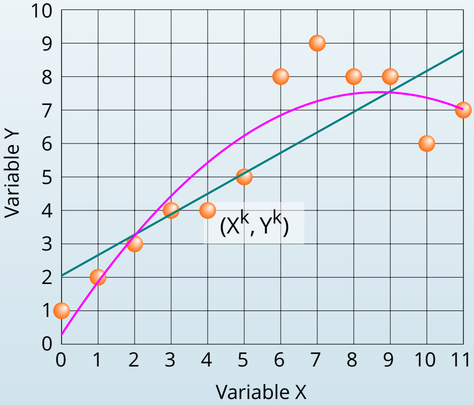
Example

- Adapt 1D metamodel to 12 data points
- Linear function



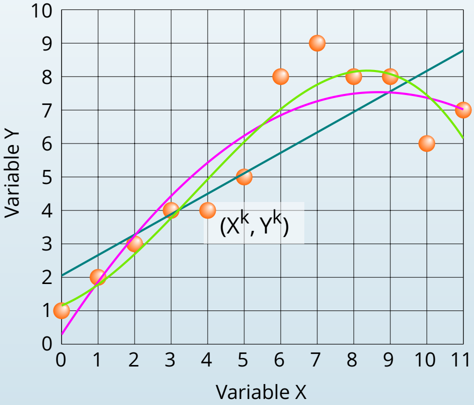
Example

- Adapt 1D metamodel to 12 data points
- Quadratic function



Example

- Adapt 1D metamodel to 12 data points
- Cubic function



Quality of metamodel

- Coeffient of determination (CoD, R^2): correlation between experimental data and model predictions

$$R^2 = \left(\frac{\mathbf{E}[(Y - \bar{Z}) \cdot (Z - \bar{Z})]}{\sigma_Y \sigma_Z} \right)^2 = \rho^2_{YZ}; Z = \sum_{i=1}^{n} p_i g_i(X)$$

- CoD may be high due to overfitting (leads to bad prediction behavior)
- Adjusted CoD for small sample sizes m (penalize overfitting)

$$R^2_{adj} = R^2 - \frac{n-1}{m-n} \left(1 - R^2\right)$$

- If an additional test data set T is available: Coefficient of Quality (CoQ)

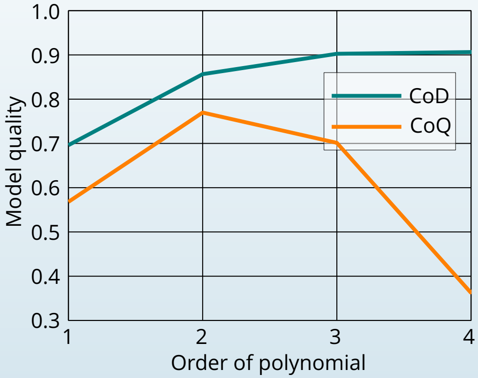
$$CoQ = \left(\frac{\mathbf{E}[(T - \bar{T}) \cdot (Z_T - \bar{Z}_T)]}{\sigma_Y \sigma_Z} \right)^2 = \rho^2_{TZ_T};\ Z_T = \sum_{i=1}^{n} p_i g_i(X_T);\ 0 \le CoQ \le 1$$

- Practical application: randomly split data into training set/test set or leave-one-out cross validation.

Previous Example

- Change model order k for best CoQ, compare to CoD

k	CoD	CoQ
1	0.70	0.57
2	0.86	0.77
3	0.90	0.70
4	0.91	0.36



Importance measures

- Several possibilities, simplest is based on linear correlations (suitable only for almost linear models)
- Suggested: Use dependence of CoQ on individual variables
- Compute CoQ for full model (all input variables)
- Remove input variable x_k from regression models, compute CoQ_k and $\Delta_k = CoQ - CoQ_k$
- Normalised importance measure $I_k = \frac{\Delta_k}{\sum \Delta_k} CoQ$
- Positive importance measures indicated important variables, negative measure indicate that variable should be removed.

Example

- 5-dimensional test function (taken from optiSLang docu)

$$g = 0.5x_1 + x_2 + 0.5x_1 x_2 + 5 \sin x_3 + 0.2x_4 + 0.1x_5$$

- All variables are in the range $[-\pi, \pi]$
- Introduce a 6th variable which does not appear in the function
- Establish DOE with 100 samples (using Latin Hypercube Sampling)
- Carry out LOO cross validation
  - Remove sample k from training data, use this as test sample
  - Adjust regression model to training data (Thin Plate Spline)
  - Apply model to test input and compute model output k
  - Repeat with next k
- Compute correlations between all test data and corresponding model outputs

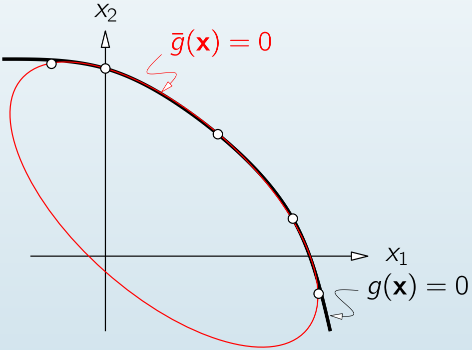
Metamodel of Optimal Quality

- Determine importance measures I_k and eliminate variables with smallest importance
- MOQ contains variables X_1, X_2, and X_3.

ρ	$I_k^{(1)}$	$I_k^{(2)}$	$I_k^{(3)}$	$I_k^{(4)}$	$I_k^{(5)}$
0.19	0.16	0.15	0.16	**0.19**	n.a.
0.46	0.30	0.30	0.25	**0.32**	0.16
0.62	0.43	0.50	0.55	**0.46**	0.44
0.06	-0.01	-0.04	-0.04	n.a.	n.a.
0.19	-0.01	-0.05	n.a.	n.a.	n.a.
-0.06	-0.06	n.a.	n.a.	n.a.	n.a.
CoQ	0.77	0.86	0.92	**0.97**	0.61

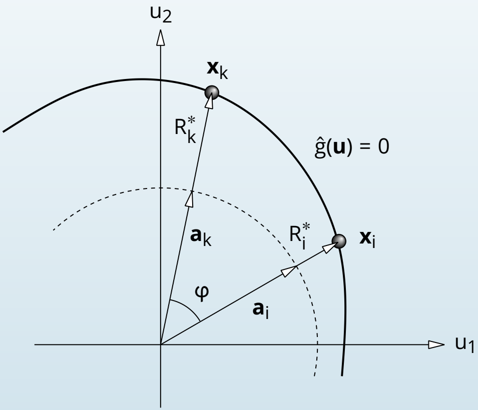
Choice of support points for reliability analysis

- Contributions to failure probability only from very narrow region near the design point
- Most important to have support points for the response surface $\hat{g}(\mathbf{x})$ very close to or exactly at the limit state $g(\mathbf{x}) = 0$



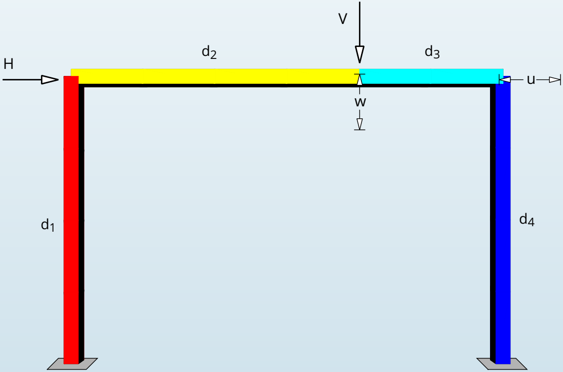
Determine support points

- Locate points on the boundary using a search procedure (e.g. bisection)
- Close similarity to directional sampling method



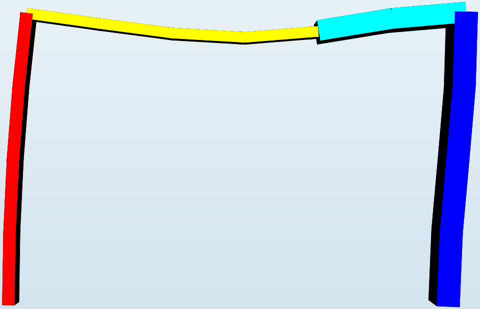
Example: Frame under static loads

- Plane frame under two static loads H and V
- Minimize structural mass subject to constraints on
  - Horizontal deflection $u < u_0$
  - Vertical deflection $w < w_0$
  - Buckling load factor $\lambda \geq \lambda_0$



Deterministic Optimization

- $E = 210$ GPa, $\rho = 7850$ kg/m^3, $H = 100$ kN, $V = 117$ kN, $u_0 = 0.05$ m, $w_0 = 0.05$ m, $\lambda_0 = 2.5$.
- Optimal cross sections (requires 100 FE analyses): $d_1 = 0.082$ m, $d_2 = 0.069$ m, $d_3 = 0.137$ m, $d_4 = 0.152$ m, $m = 1388$ kg.
- Deformed optimal structure (magnified 5x)



Stochastic Problem

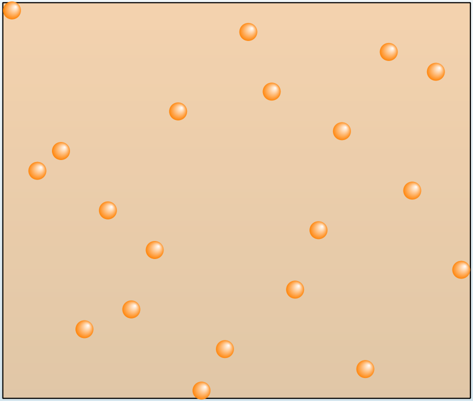
- Loads are random variables with mean values $\bar{H} = 100$ kN, $\bar{V} = 117$ kN and coefficients of variation of 5%.
- Constraints are satisfied with prescribed reliability levels $\beta_u = \beta_w = 3$, $\beta_\lambda = 4$.
- Two approaches
  - Method of safety factors: Upscale deterministic optimum cross sections such as to satisfy probabilistic constraints
    Leads to design with mass m = 1706 kg (increase of 23%).
  - Stochastic optimization (RBDO): Include probabilistic constraints into the optimization process

Stochastic Optimization

- Loads are random variables with mean values $\bar{H} = 100$ kN, $\bar{V} = 117$ kN and coefficients of variation of 5%.
- Constraints are satisfied with prescribed reliability levels $\beta_u = \beta_w = 3$, $\beta_\lambda = 4$.
- Probabilities of constraint violation computed by FORM
- Straightforward analysis requires about 35.000 structural analyses.
- Optimal cross sections:
  $d_1 = 0.081$ m, $d_2 = 0.076$ m, $d_3 = 0.150$ m, $d_4 = 0.171$ m
  m = 1657 kg (19% increase).

Adaptive Response Surface Method (ARSM)

- Repeated application of DOE scheme based on previous optimization results



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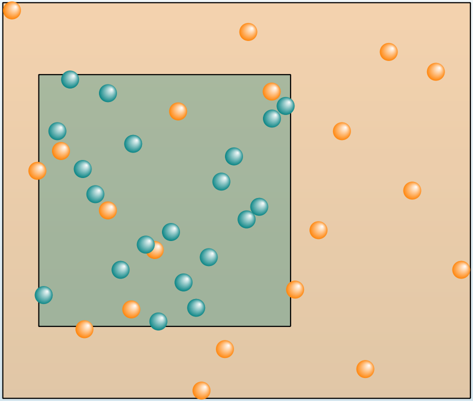

Adaptive Response Surface Method (ARSM)

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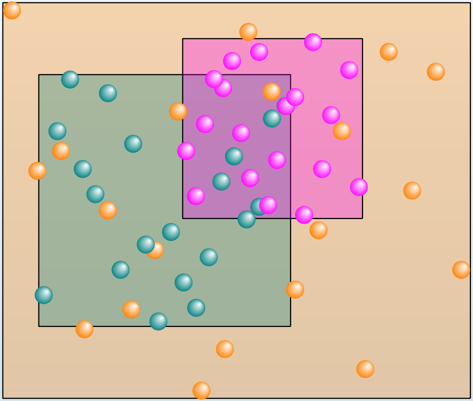
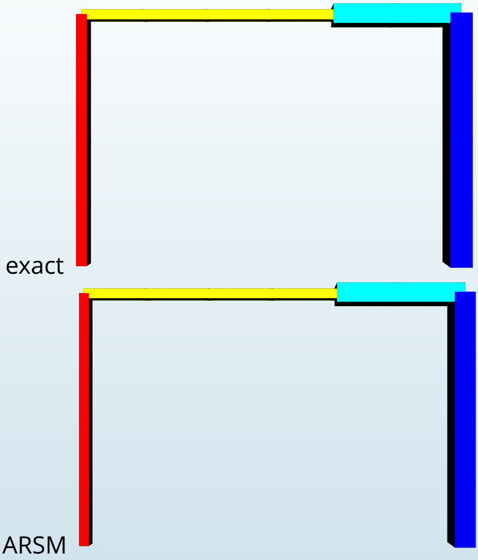


Application of ARSM 1

1. Initial DOE with 256 structural analyses
2. Approximate constraint functions by Metamodels of Optimal Quality
3. Carry out stochastic optimization
4. re-center DOE and narrow range (factor 0.7), loop to Step 2 or break
5. Check feasibility of approximate solution

- 4 iterations result in
  $d_1 = 0.083$ m, $d_2 = 0.078$ m, $d_3 = 0.144$ m, $d_4 = 0.172$ m
  $m = 1665$ kg.
- Result is not feasible (Constraint 1 is slightly violated)
- Upscale solution by 0.2% satisfies all constraints, $m = 1673$ kg (20.5%) increase.
- Compared to full stochastic analysis: reduce computation by factor 35.
- Compared to deterministic analysis: increase factor by 10.

Application of ARSM 2



Further reading

- C. Bucher: Computational Analysis of Randomness in Structural Mechanics, Taylor & Francis, 2009.