Stochastic Analysis in Structural Optimization

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Zusammenfassung

Structural optimization typically aims at high performance levels for a clearly specified set of conditions. Unfortunately, this goal can usually be achieved only by sacrificing robustness of the design. This implies a high sensitivity with respect to unforeseen stochastic situations or unavoidable random manufacturing tolerances. In order to prevent structural failure due to loss of robustness it is therefore desirable to incorporate a suitable measure of robustness into the optimization process. This can be achieved by introducing additional constraint conditions or appropriate modifications of the objective function. An example for such a design concept is *reliability-based optimization* based on the notion of the failure probability. This is most appropriate for high-risk structures such as e.g. power-generating facilites. Alternatively, simpler stochastic measures such as variances or standard deviations might be more appropriate for the design of low-risk structural elements which are frequently found e.g. in the automotive industry. The paper discusses the basic requirement for robust optimization and attempts to outline pros and cons of different approaches to the solution of this problem.

Keywords: Robustness, reliability, stochastics, optimization

1 INTRODUCTION

Uncertainties in the optimization process can be attributed to three major sources as shown in Fig. 1 These sources of uncertainties or stochastic scatter are

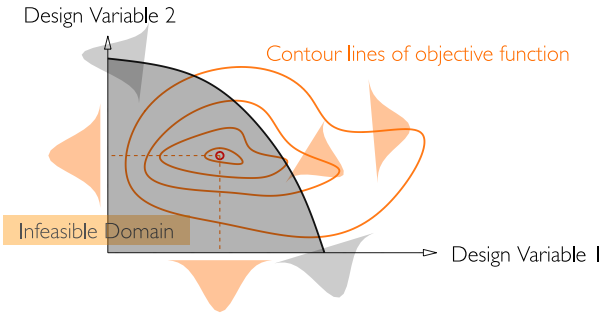


Figure 1: Sources of uncertainty in optimization

- Uncertainty of design variables. This means that the manufacturing process is unable to achieve the design precisely. The magnitude of such uncertainty depends to a large extent on the quality control of the manufacturing process.
- Uncertainty in the objective function. This means that some parameters affecting the structural performance are beyond the control of the designer. These uncertainties may be reduced by a stringent specification of operating conditions. This may be possible for mechanical structures, but is typically not feasible for civil structures subjected to environmental loading such as earthquakes or severe storms which cannot be controlled.
- Uncertainty of the feasible domain. This means that the admissibility of a particular design (such as its safety or serviceability) cannot be determined deterministically. Such problems are at the core of probability-based design of structures.

2 STOCHASTIC MODELING

2.1 Basic Definitions

Probability in the mathematical sense is defined as a positive measure (between 0 and 1) associated with an event in probability space. For most physical phenomena this event is suitably defined by the occurrence of a real-valued random value X which is smaller than a prescribed, deterministic value x. The probability associated with this event is called *probability distribution function* (or, equivalently *cumulative distribution function*, cdf):

$$F_X(x) = P[X < x] \tag{1}$$

Differentiation of $F_X(x)$ with respect to x yields the so-called *probability density function* (pdf):

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{2}$$

A qualitative representation of these functions is given in Fig. 2.
In many cases it is convenient to characterize random variables in terms of expected values rather than probability density functions. Special cases of expected values are the *mean value* $\bar{X}$:

$$\bar{X} = \mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{3}$$

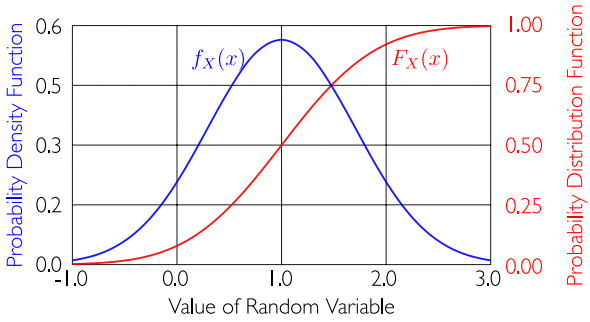


Figure 2: Schematic sketch of probability distribution and probability density functions.

and the *variance* σ_X^2 of a random variable:

$$\sigma_X^2 = \mathbf{E}[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx \tag{4}$$

The positive square root of the variance σ_X is called *standard deviation*. For variables with non-zero mean value ($\bar{X} \neq 0$) it is useful to define the dimension-less coefficient of variation

$$V_X = \frac{\sigma_X}{\bar{X}} \tag{5}$$

A description of random variables in terms of mean value and standard deviation is sometimes called "second moment representation". Note that the mathematical expectations as defined here are so-called *ensemble averages*, i.e. averages over all possible realizations.

2.2 Two Types of Distributions

Due to its simplicity, the so-called Gaussian or normal distribution is frequently used. A random variable X is *normally distributed*, if its probability density function is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x - \bar{X})^2}{2\sigma_X^2}\right]; \quad -\infty < x < \infty \tag{6}$$

Here $\bar{X}$ is the mean value, and σ_X is the standard deviation. The distribution function $F_X(x)$ is described by the normal integral $\Phi(.)$:

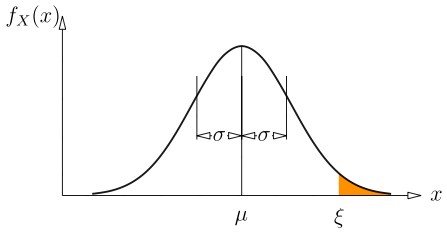
$$F_X(x) = \Phi(\frac{x - \bar{X}}{\sigma_X}) \tag{7}$$

in which

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-\frac{u^2}{2}) du \tag{8}$$

This integral is not solvable in closed form, however tables and convenient numerical approximations exist. The use of the Gaussian distribution is frequently motivated by the central limit theorem which states that an additive superposition of independent random effects tends asymptotically to the Gaussian distribution.
A random variable X is *log-normally distributed*, if its pdf is:

$$f_X(x) = \frac{1}{x\sqrt{2\pi}s} \exp\left[-\frac{(\log \frac{x}{\mu})^2)}{2s^2}\right]; \quad 0 \leq x < \infty \tag{9}$$



$$P_\xi = P[X \geq \xi]$$

ξ	μ	$\mu + \sigma$	$\mu + 2\sigma$	$\mu + 3\sigma$	$\mu + 4\sigma$
P_ξ	$5 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$2.3 \cdot 10^{-2}$	$1.4 \cdot 10^{-3}$	$3.2 \cdot 10^{-5}$

Figure 3: Gaussian (normal) probability density function and probabilites of exceeding threshold values ξ

and its distribution function is given by

$$F_X(x) = \Phi(\frac{\log \frac{x}{\mu}}{s}) \tag{10}$$

In these equations, the parameters μ and s are related to the mean value and the standard deviation as follows:

$$\mu = \bar{X} \exp(-\frac{s^2}{2}); \quad s = \sqrt{\ln(\frac{\sigma_X^2}{\bar{X}^2} + 1)} \tag{11}$$

Two random variables with $\bar{X} = 1.0$ and $\sigma_X = 0.5$ having different distribution types are shown in Fig. 4. It is clearly seen that the log-normal density function is non-symmetric and does not allow negative values. Another important difference lies in the fact that the probability

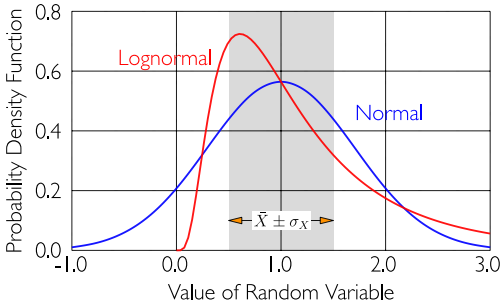


Figure 4: Normal and log-normal probability density functions.

of exceeding certain threshold levels ξ is significantly influenced by the type of probability distribution. For a normal distribution, the probability of exceeding a level $\xi = 3$ corresponding to the mean value plus 4 standard deviations is $3.2 \cdot 10^{-5}$ while in the case of a lognormal distribution the same threshold has an exceedance probability of 0.083. In order to achieve the same exceedance probability as in the Gaussian case, the threshold level must be set to $\xi = 7.39$, which is the mean value plus 12 standard deviations.

2.3 Random Vectors

In many applications a large number of random variables occur together. It is conceptually helpful to assemble all these random variables $X_k;\ k = 1 \ldots n$ into a *random vector* $\mathbf{X}$:

$$\mathbf{X} = [X_1, X_2, \ldots X_n]^T \tag{12}$$

For this vector, expected values can be defined in terms of expected values for all of its components:

Mean value vector

$$\bar{\mathbf{X}} = \mathbf{E}[(\mathbf{X}] = [\bar{X}_1, \bar{X}_2, \ldots \bar{X}_n]^T \tag{13}$$

Covariance matrix

$$\mathbf{E}[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^T] = \mathbf{C_{XX}} \tag{14}$$

The dimensionless quantity

$$\rho_{ik} = \frac{\mathbf{E}[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]}{\sigma_{X_i}\sigma_{X_k}} \tag{15}$$

is called *coefficient of correlation*. Its value is bounded in the interval $[-1,\ 1]$.

3 ANALYSIS OF RESPONSE VARIABILITY

3.1 General Remarks

In order to obtain meaningful correlations between the input and output variables it is essential to precisely capture the input correlations in the simulated values. Monte-Carlo based methods use digital generation of pseudo-random numbers to produce artificial sample values for the input variables. The quality of these numbers can be measured in terms of their statistical properties. For the case of two random variables X_1 and X_2, Monte Carlo methods produce sequences of numbers $X_1^k, X_2^k, k = 1 \ldots N$ in such a way that the prescribed statistics as estimated from these samples match the prescribed statistics as closely as possible. Typically, plain Mont-Carlo methods are fairly well able to represent individual statistics of the random variables. At small sample sizes N, however, the prescribed correlation structure may be rather heavily distorted. Significant improvement can be made by utilizing the Latin Hypercube sampling method Florian (1992). Comparing Plain Monte-Carlo (PMC) with Latin Hypercube Sampling (LHS) it is easily seen that LHS covers the space of random variables in a significantly superior way. In particular, PMC introduces unwanted correlation into the samples which becomes very pronounced if the number of samples is small. This is readily seen from Fig. 5, where a positive correlation of $\rho = 0.28$ appears in the samples on the left hand side. Unfortunately, many real-world structural problem are so large that only a small number of samples can be accepted.

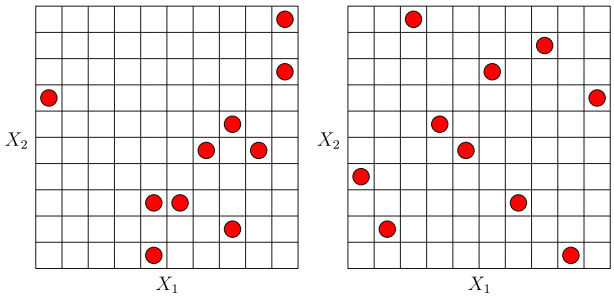


Figure 5: 10 samples of two uniformly distributed independent random variables. Left: Plain Monte Carlo, Right: Latin Hypercube Sampling

3.2 Correlation Statistics

Assume that we want to estimate a matrix of correlation coefficients of m variables from N samples. This matrix has $M = m \cdot (m-1)/2$ different entries in addition to m unit elements on

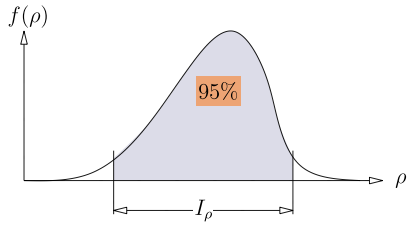


Figure 6: Confidence interval for estimated coefficients of correlation ρ

the main diagonal. The confidence intervals for the estimated coefficients of correlation ρ_{ij} are computed based on the Fisher's z-transformation. The interval for a significance level of α (i.e. a confidence level of $1-\alpha$) is given by

$$[\tanh(z_{ij} - \frac{z_c}{\sqrt{N-3}}),\ \tanh(z_{ij} + \frac{z_c}{\sqrt{N-3}})] \tag{16}$$

In this equation, N ist he number of samples used for the estimation of ρ_{ij}. The critical value z_c is computed by using the Bonferroni-corrected value for the significance level $\alpha' = \alpha/M$ with M being the number of correlation coefficients to be estimated (see above). The transformed variable z is computed from

$$z_{ij} = \frac{1}{2} \log \frac{1+\rho_{ij}}{1-\rho_{ij}} \tag{17}$$

and the critical value z_c is given by

$$z_c = \Phi^{-1}(1-\alpha'/2) \tag{18}$$

where $\Phi^{-1}(.)$ is the inverse cumulative Gaussian distribution function.

3.3 Effect of Latin Hypercube sampling

In order to study the effect of LHS on the reduction of statistical uncertainty, a numerical study performing a comparison of the estimation errors (standard deviations) of the correlation coefficients is carried out. The following table shows confidence interval for a confidence level of 95% as a function of the correlation coefficient ρ and the number of samples N used for one estimation. The statistical analysis is repeated 1000 times. In summary, it turns out that the net effect of LHS is an effective reduction of the sample size by a factor of more than 10. For example, as seen from Tables 1 and 2, it is possible to estimate a coefficient of correlation of $\rho = 0.3$ using 1000 samples of MCS with a 95%-confidence interval of 0.11, while the same confidence interval (actually 0.1) is achieved with only 100 samples using LHS. On the other hand, 1000 LHS samples would reduce the respective 95%-confidence interval to 0.03, which is an enormous improvement.

Table 1: 95% confidence interval of correlation coefficient, Plain Monte Carlo

	ρ				
N	0	0.3	0.5	0.7	0.9
10	1.261	1.231	1.054	0.757	0.299
30	0.712	0.682	0.557	0.381	0.149
100	0.409	0.374	0.306	0.199	0.079
300	0.230	0.209	0.170	0.116	0.045
1000	0.124	0.115	0.093	0.062	0.023

Table 2: 95% confidence interval of correlation coefficient, Latin Hypercube Sampling

N	ρ				
	0	0.3	0.5	0.7	0.9
10	0.420	0.382	0.260	0.158	0.035
30	0.197	0.194	0.139	0.073	0.018
100	0.111	0.101	0.071	0.042	0.009
300	0.065	0.057	0.042	0.024	0.006
1000	0.038	0.033	0.025	0.014	0.003

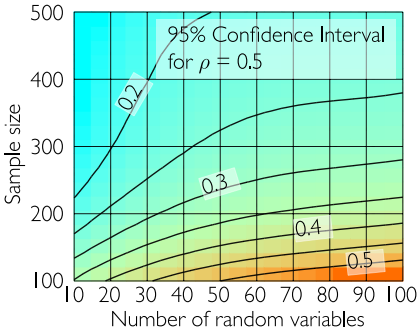


Figure 7: Confidence intervals for coefficients of correlation

4 RELIABILITY ANALYSIS

4.1 Definition

Generally, failure (i.e. an undesired or unsafe state of the structure) is defined in terms of a limit state function $g(.)$ defining the set $\mathcal{F} = \{\mathbf{X} : g(\mathbf{X}) \leq 0\}$. Frequently, $Z = g(\mathbf{X})$ is called *safety margin*. The failure probability is defined as the probability of the occurrence of $\mathcal{F}$:

$$P(\mathcal{F}) = P[\{\mathbf{X} : g(\mathbf{X}) \leq 0\}] \tag{19}$$

4.2 FORM - First Order Reliability Method

The FORM-Concept is based on a description of the reliability problem in standard Gaussian space Rackwitz und Fiessler (1978). Hence transformations from correlated non-Gaussian variables $\mathbf{X}$ to uncorrelated Gaussian variables $\mathbf{U}$ with zero mean and unit variance are required. This concept is especially useful in conjunction with the Nataf-model for the joint pdf of $\mathbf{X}$ Liu und DerKiureghian (1986). Eventually, this leads to a representation of the limit state function $g(.)$ in terms of the standardized Gaussian variables U_i:

$$g(\mathbf{X}) = g(X_1, X_2, \ldots X_n) = g[X_1(U_1, \ldots U_n) \ldots X_n(U_1, \ldots U_n)] \tag{20}$$

This function is linearized with respect to the components in the expansion point $\mathbf{u}^*$. This point is chosen to minimize the distance from the origin in Gaussian space. From this geometrical interpretation it becomes quite clear that the calculation of the design point can be reduced to an optimization problem:

$$\mathbf{u}^* : \mathbf{u}^T\mathbf{u} \rightarrow \text{Min.}; \quad \text{subject to: } g[\mathbf{x}(\mathbf{u})] = 0 \tag{21}$$

Standard optimization procedures can be utilized to solve for the location of $\mathbf{u}^*$ Shinozuka (1983). In the next step, the exact limit state function $g(\mathbf{u})$ is replaced by a linear approximation $\bar{g}(\mathbf{u})$ as shown in Fig. 8. From this, the probability of failure is easily determined to be

$$P(\mathcal{F}) = \Phi(-\beta) \tag{22}$$

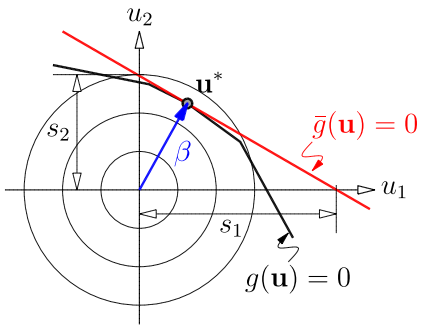


Figure 8: Linearization required for first order reliability method

5 APPLICATION EXAMPLE

As an example, consider the weight optimization of a simple beam subjected to a dynamic loading (cf. Fig. 9). For this beam with a rectangular cross section (w, h) subjected to a harmonic loading $F(t)$ the mass should be minimized considering the constraints that the center deflection due to the loading should be smaller than 10 mm. Large deflections are considered to be serviceability failures. The design variables are bounded in the range $0 < w, h < 1$.

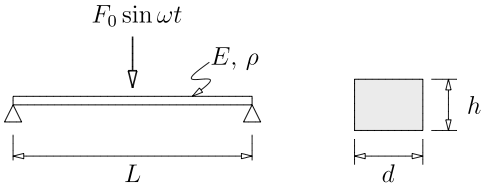


Figure 9: Beam with rectangular cross section

First, the problem is formulated in terms of deterministic parameters. It turns out that the feasible domain is not simply connected (cf. Fig. 10). Such problems typically exhibit multiple local minima. It is then highly recommendable to apply a non-local search strategy for the optimization.
For numerical values of $F_0 = 20\,kN$, $\omega = 60\,rad/s$, $E = 3{\cdot}10^{10}\,N/m^2$, $\rho = 2500\,kg/m^3$, $L = 10\,m$ and $g = 9.81\,m/s^2$ the objective function (i.e. the cross sectional area) and the feasible domain are shown in Fig. 10. It can be seen that there are two separate parts of the feasible domain. It is well known that gradient-based optimization techniques have difficulties crossing domain boundaries and localizing the global minimum. The global minimum is located at $w = 0.06$ and $h = 1.00$ as indicated in Fig. 10.

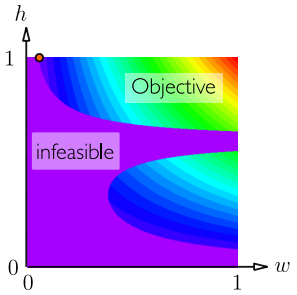


Figure 10: Objective function and feasible domain, deterministic situation

In the next step, the loading amplitude F_0 and the excitation frequency ω are assumed to be

Gaussian random variables. The mean values are assumed to be the nominal values as given above, and the coefficients of variation are assumed to be of the order of 10%. This implies that the constraints can be satisfied only with a certain probability < 1. In addition to that, stochastic uncertainty in the design variables w and h is considered as well. For this, it is assumed that the optimization controls the mean values of $\bar{w}$ and $\bar{h}$, and that the actual structural dimensions are log-normally distributed random variables with at coefficient of 10%. Fig. 11 shows the probability $P(\mathcal{F}|\bar{w},\bar{h})$of violating the constraint as a function of the design variables $\bar{w}$ and $\bar{h}$.

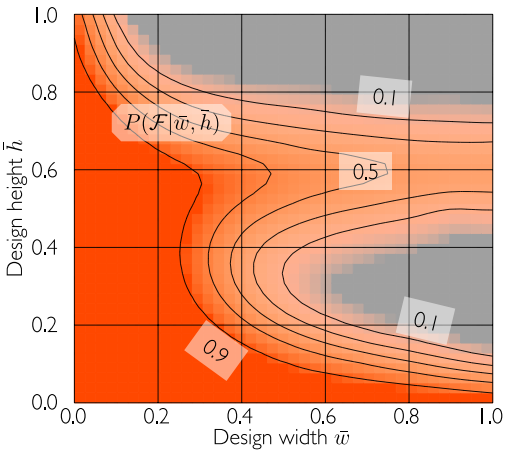


Figure 11: Conditional failure probability $P(\mathcal{F}|\bar{w},\bar{h})$ depending on $\bar{w}$ und $\bar{h}$

For the following optimization the constraint is formulated as the condition that the probability of violating the prescribed displacement threshold of 10 mm should be smaller than 1%. In the context of a genetic optimization algorithm, constraints are frequently formulated in terms of a penalty function added to the objective function. The magnitude of the penalty term is chosen to depend on the magnitude of the failure probability. The total objective function then becomes

$$L = \bar{h} \cdot \bar{w} + S \cdot H[P(\mathcal{F}) - 0.01] \cdot [P(\mathcal{F}) - 0.01] \tag{23}$$

In this equation, $H[.]$ is the Heaviside function which yields zero penalty for feasible designs. The penalty scale is assumed to have the value of $S = 100$. An optimization run with 10 generation, each having 100 individuals yielded the best individual at $w = 0.888$ and $h = 0.289$. The probability of failure was $P(\mathcal{F}) = 0.0098$, i.e. close to the acceptable limit of 0.01. The cross sectional area was 0.26 which is considerably larger than in the value of 0.06 obtained in the deterministic case.
Fig. 12 shows the progress of genetic optimization. The first, third, fifth, and 10-th generations are shown. The concentration of the populations in the region of acceptable probability of failure is easily seen.
One important outcome of this example is the fact that the locations of the deterministic optimization and the probability-based robust optimization are entirely different. This emphasizes the necessity of incorporation robustness and reliability analysis into the optimization process.

6 CONCLUDING REMARKS

Structural optimization tends to lead to highly specialized designs which, unfortunately, very likely lack robustness of performance with respect to unforeseen situations. A prominent cause for such situation lies in the inherent randomness of either design parameters or constraint conditions. One possible way to overcome this dilemma lies in the application of robustness-based optimization. This allows to take into account random variability in the problem formulation thus leading to optimal designs which are automatically robust. It appears that this concept should

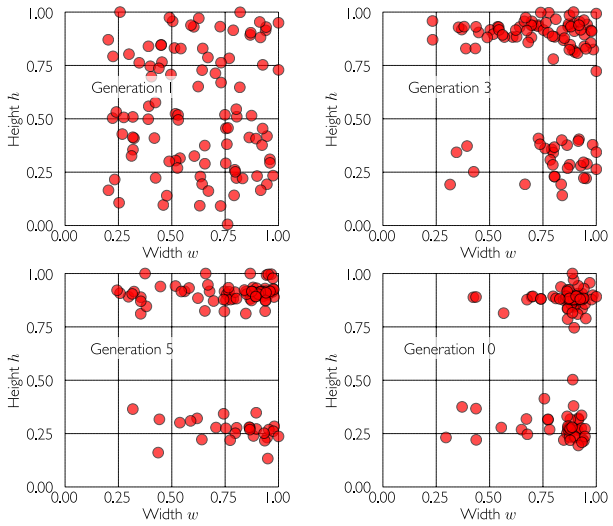


Figure 12: Progress in genetic optimization

be applicable to a large number of structural optimization problems. However, the numerical effort to carry out the analysis is quite substantial. Further research into the reduction of effort is therefore required.

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