Non-parametric Structural Reliability Analysis using Random Fields and Robustness Evaluation

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Abstract

In reliability and robustness analysis, imperfections of a mechanical or structural system, such as material properties or geometrical deviations, are modelled as random fields in order to account for their fluctuations over space. A random field normally comprises a huge number of random variables. The present paper proposes a method to reduce the random variables set. This reduction is performed on the basis on a robustness analysis. In this way, numerical difficulties can be avoided and the efficiency of the subsequent reliability analysis is enhanced.

As an example, the reliability of a cylindricral shell structure with random imperfections is studied. Within this example, the imperfections are discretized by Stochastic Finite Element methods. it is demonstrated, how robustness analysis is employed in order to identify the most relevant random variables. The probability of failure is computed by Monte Carlo simulation involving Latin Hypercube sampling. The failure criterion is derived from a comparison of the linear buckling loads of the perfect and the imperfect structures.

This so-called non-parametric structural reliability analysis is a new method to estimate the safety and reliability of finite element structures in such cases where a CAD-based parametrization is not possible or not meaningful. The probabilistic and structural analysis tasks are performed with the optiSLang, SoS and SLang software packages.

Keywords: Random field, imperfection, stochastic finite elements, geometrical nonlinearity, buckling, stability.

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1 Introduction

A realistic interpretation of the computational analysis of a mechanical or structural system, in particular when compared to measurements, requires to take into account the natural scatter of the structural properties. This uncertainty can be conveniently modelled with methods of probability calculus. I.e., the properties are characterized by distribution types and statistic moments. It is a major goal of stochastic computational analysis to relate the uncertainties of the input variables to the uncertainty of the model response performance. Several levels of stochastic modelling can be named as:

- Time-independent random problems (Reliability analysis)
  - Random variables (constant in time and space),
  - Random fields (correlated fluctuations in space),
- Time-dependent random problems (First passage reliability analysis)
  - Random processes (correlated fluctuations in time).

The stochastic analysis software SLang (which stands for: Structural Language) includes several methods to solve all of the above stochastic models. Currently, optiSLang (the optimizing Structural Language) supports methods to analyse random variables only. It offers additional functions for robustness evaluation and structural optimization. In addition, the SoS (Statistics on Structures) add-on tool to optiSLang provides methods to solve random fields.

There exist attempts to model uncertainties by few random variables which are generated by CAD programs. This approach is meaningful for few special problems only, since the mentioned fluctuations in time and/or space cannot be modelled in this way. Random processes or fields need to be discretized for a computer implementation. The required number of random variables, however, can be considerably high. The present paper proposes a method to model a random field with a reduced set of variables. Robustness evaluation (section 3) is employed for this purpose, which relates the stochastic input and output quantities and thus helps to identify the relevant variables. Unlike previous approaches (sketched in sec. 2.2), it does not rely on purely stochastic considerations, but takes into account the structural behaviour as well.

2 Random Fields

2.1 Properties

A random field is, in brief, a random function $H(\mathbf{r})$ defined on a spatial structure. The vector $\mathbf{r} \in \mathbb{R}_{Str.}$ points to a location on the structure. Random fields are used, e.g., to study random fluctuations in geometrical or material properties of a mechanical or structural system. In other words, the considered property is a random variable at each point on the structure. Moreover, the random properties at two different locations can be mutually correlated among each others.

Any random variable is characterized by a probability distribution function, which can be parameterized by distribution type and stochastic moments. For random fields, the

moments become functions over space as well. *From now on, a Gaussian (or Normal) distribution type is assumed.* In this case, the characterization by first and second moments provides the full information. In particular

$$\mu_H(\mathbf{r}) = \mathrm{E}[H(\mathbf{r})] = \int_{-\infty}^{+\infty} h f_H(\mathbf{r}, h)\ dh \tag{1}$$

denotes the *mean function*, and

$$R_{HH}(\mathbf{r}_1, \mathbf{r}_2) = \mathrm{E}[H(\mathbf{r}_1) \cdot H(\mathbf{r}_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1 h_2 f_H(\mathbf{r}_1, \mathbf{r}_2, h_1, h_2)\ dh_1\ dh_2 \tag{2}$$

the *correlation function*, with E[.] being the expected value operation (see Soong and Grigoriu, 1993). R_{HH} is a function of the distance between two points and indicates the amount of linear dependency between the random properties at these locations. It has the properties of symmetry:

$$R_{HH}(\mathbf{r}_1, \mathbf{r}_2) = R_{HH}(\mathbf{r}_2, \mathbf{r}_1) \tag{3}$$

and positive semi-definiteness:

$$\iint_{\mathbb{R}_x\mathbb{R}_x} R_{HH}(\mathbf{r}_1, \mathbf{r}_2) w(\mathbf{r}_1) w(\mathbf{r}_2)\ d\mathbf{r}_1\ d\mathbf{r}_2 \geq 0 \tag{4}$$

for any real valued function $w(\mathbf{r} \in \mathbb{R}_{Str.})$ defined on the structure.

The so-called *correlation length* L_{HH}, which is actually the centre of gravity of the correlation function, is a typical characteristic of R_{HH}. It has to be estimated from manufacturing processes, natural scatter of material properties, etc. An infinite correlation length yields a structure with random properties, yet without fluctuations within the structure. A zero correlation length yields uncorrelated (in case of the Gaussian distribution independent) variables.

Two special cases are important for the further studies. *Homogeneity:* A random field is said to be homogeneous in the wide sense, if the first and second moments are the same at any possible location, i.e.

$$\mu_H(\mathbf{r}) = \mu_H(\mathbf{r} + \boldsymbol{\xi}) \quad \forall\, \boldsymbol{\xi} \tag{5}$$

$$R_{HH}(\mathbf{r}_1, \mathbf{r}_2) = R_{HH}(\mathbf{r}_1 + \boldsymbol{\xi}, \mathbf{r}_2 + \boldsymbol{\xi}) \quad \forall\, \boldsymbol{\xi} \tag{6}$$

Isotropy (in the wide sense) claims that the correlation function depends on the distance between the two observed locations $\mathbf{r}_1,\ \mathbf{r}_2$ only, not on the direction:

$$R_{HH}(\mathbf{r}, \mathbf{r} + \boldsymbol{\xi}) = R_{HH}(\|\boldsymbol{\xi}\|) \tag{7}$$

2.2 Modelling

For computational analyses, a random field has to be discretized in order to yield a finite set of random variables $\mathbf{X}$, which are assigned to discrete locations on the observed

structure. Since the Finite Element Method is the standard for structural analyses, it is convenient to discretize the random field in the same way as the finite element model. One speaks of Stochastic Finite Elements in this case. The discretization can be oriented at the element mid points, integration points, or nodes. The properties of the random variables are derived from the random field properties explained previously. The spatial discretization should be able to model the variability of the random field. For this purpose, it has been recommended by Der Kiureghian and Ke (1988); Hisada and Nakagiri (1981) that the distance between two discretization points should be not more than 1/4 of L_{HH}.

The set of random variables is then characterized by a mean vector and a correlation matrix. It is convenient for the developments that follow to use the covariance matrix instead, which is defined as

$$\mathbf{C}_{XX}: \quad c_{ij} = R_{HH}(\mathbf{r}_i, \mathbf{r}_j) - \mu_H(\mathbf{r}_i) \cdot \mu_H(\mathbf{r}_j) \tag{8}$$

The joint density of all random variables can be modelled with help of the Nataf model (Nataf, 1962; Liu and Der Kiureghian, 1986), given the type and properties of the marginal distributions for each variable.

From now on, *random fields with zero mean vector are considered.* Then the covariance matrix suffices for the characterization of the random variables set. Random number generators can produce independent random variables only. For the assumed case of Gaussian distributed variables, independence is equivalent to zero correlation. It can be shown that the random variables will be uncorrelated after the following transformation. The covariance matrix is decomposed with help of an eigenvalue analysis:

$$\mathbf{\Psi}^T \mathbf{C}_{XX} \mathbf{\Psi} = \operatorname{diag}\{\lambda_i\} \tag{9}$$

Therein, $\mathbf{\Psi}$ is the matrix of eigenvectors of $\mathbf{C}_{XX}$ stored columnwise, and the eigenvalues are identical to the variances of the uncorrelated random variables Y_i: $\lambda_i = \sigma^2_{Y_i}$. The transformation rule reads

$$\mathbf{Y} = \mathbf{\Psi}^T \mathbf{X} \tag{10}$$

and the backward transformation

$$\mathbf{X} = \mathbf{\Psi} \ \ \mathbf{Y} \tag{11}$$

because the eigenvectors $\mathbf{\Psi}$ form an orthonormal basis. Hence it is possible to simulate the random field with a set of uncorrelated random variables $\mathbf{Y}$ and the respective (deterministic) shape funtions $\mathbf{\Psi}$.

The eigenvalues are usually stored sorted by magnitude in descending order, which is a measure for their relevance in representing $\mathbf{C}_{XX}$. This opens a way of reducing the usually huge number of variables. Only the random variables with the highest variances are needed for simulation. The quality of approximation of the random field is expressed by the variability fraction (Brenner, 1995)

$$Q = \frac{\sum_{i=1}^{n} \sigma^2_{Y_i}}{\operatorname{trace}\left(\mathbf{C}_{XX}\right)} \ ; \quad 0 \le Q \le 1 \tag{12}$$

The number of the random variables considered has to be adjusted before the simulation in order to reach a sufficient quality, e.g. $Q > 0.9$.

3 Robustness Analysis

The number of random variables used to model a random field can be very high, such that numerical difficulties may occur, particularly when the probability of failure is low. Experience with other applications of reliability analysis with geometrical imperfection (e.g. in Schorling, 1998) shows, that the reduction of variables explained in section 2, eq. (12) fails, because the criterion is purely stochastic and does not account for the structural behaviour. On the other hand, purely mechanical considerations may not work as well, if they fail to represent the random field properly.

The program optiSLang (DYNARDO GmbH, 2006) offers the feature of robustness analysis. This is used to find a suitable selection of variables. In brief, the robustness analysis examines statistical dependencies between any input and output quantities the user desires. The data is obtained from a Monte Carlo simulation with small sample size. The input variables are simulated following statistical properties provided by the user, or they are varied systematically within given bounds. optiSLang comprises functions such as filters, fit tests or a principal component analysis of the correlation matrix, which shall reveal the most relevant influences on the output quantities observed. Results are plotted as coloured matrices, histograms etc. which make it easy to identify dependencies between variables. Two functions are explained in more detail in the following.

By computing the *quadratic correlation* it is tested, if one variable Y can be represented by a quadratic regression of another variable, X. The regression model is

$$\widehat{Y}(X) = a + bX + cX^2 \tag{13}$$

Samples of $\widehat{Y}$ are gained by inserting samples of X into eq. (13), values of Y itself are computed directly. Then the correlation coefficients $\rho_{Y\widehat{Y}}$ and $\rho_{\widehat{Y}Y} \neq \rho_{Y\widehat{Y}}$ are evaluated. The values range from 0 to 1, high values indicate a strong quadratic correlation between X and Y.

The *coefficient of determination* is the ratio of variances of a regression model and the original variable. It indicates the amount of variability of an output variable Y, which can be explained by the variability of the input variable X underlying the regression model. For the quadratic regression of eq. (13):

$$R^2 = \frac{\sum_{i=1}^{n} \left(\widehat{Y}_i(X_i) - \mu_Y\right)^2}{\sum_{i=1}^{n} \left(Y_i - \mu_Y\right)^2} \tag{14}$$

Values of R^2 vary from 0 to 1. A high value, e.g. $R^2 = 0.8$, means that 80 % of the variability of Y can be explained by a quadratic relation between Y and X. However, this is no accuracy measure of the regression model. While the (quadratic) correlation coefficients only give information about the mutual relation of two variables, the coefficients of determination allow for a comparison of the influences of all input variables to the output.

4 Example: Cylindrical Shell

4.1 Structure and Random Field Properties

The concept of non–parametric reliability analysis shall be demonstrated by an application example. The structure considered is a cylindrical shell with properties as given in table 1. The cylinder has a Navier–type suppport along the top and bottom edges and is loaded along the top edge with a continuous vertical unit load. The structure is modelled with a 9-node isoparametric shell finite element type within the program SLang (Institut für Strukturmechanik, Bauhaus-Universität). Figure 1 is a sketch of the finite element model with loads and restraints.

A random field is applied on the structure in order to model geometrical imperfections. The random properties are coordinate deviations from the perfect structure in the cylinder's radial direction. Thus the random field is discretized at the nodes of the finite element mesh. It has zero mean and a standard deviation of $\sigma_H = 10^{-3}$ mm, which is roughly a hundredth of the radius. The orthogonal field has different correlation functions along the perimeter and height as plotted in fig. 2. The spatial correlation structure with respect to one node on the bottom edge is visualized, too. While fig. 3 shows a few eigenvectors of the covariance matrix, a sampled imperfection shape can be seen in fig. 4, both are scaled.

Table 1: Properties of cylindrical shell structure.

Wall thickn. [mm]	Radius [mm]	Height [mm]	Young's mod. [N/mm^2]
0.197	101.6	139.7	$6.895 \cdot 10^4$

4.2 Stability Analysis

The reliability of the imperfect structure towards stability failure shall be studied here. The method of analysis to be applied has to be a compromise between accuracy and computing time. For the example observed here, which shows a pre–buckling behaviour close to linear and a sudden buckling failure, the linear buckling analysis suffices. It is very fast and not prone to being trapped on a postbuckling equilibrium state. The method is explained briefly in the following.

First, the initial stiffness matrix of the unloaded structure is established, denoted as ${}^0_0\mathbf{K}$. After applying a small initial load, the secant matrix ${}^t_0\mathbf{K} - {}^0_0\mathbf{K}$ is evaluated. Failure occurs when

$$\det\left({}^0_0\mathbf{K} + \lambda({}^t_0\mathbf{K} - {}^0_0\mathbf{K})\right) = 0 \tag{15}$$

Determination of the buckling load can be formulated as the generalized eigenvalue problem

$${}^0_0\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\lambda}\left({}^0_0\mathbf{K} - {}^t_0\mathbf{K}\right)\boldsymbol{\Phi} \tag{16}$$

In the above, the eigenvalues stored in $\boldsymbol{\lambda}$ are incrementation factors of the initial load which lead to buckling failure. The smallest value λ_1 is the relevant one.

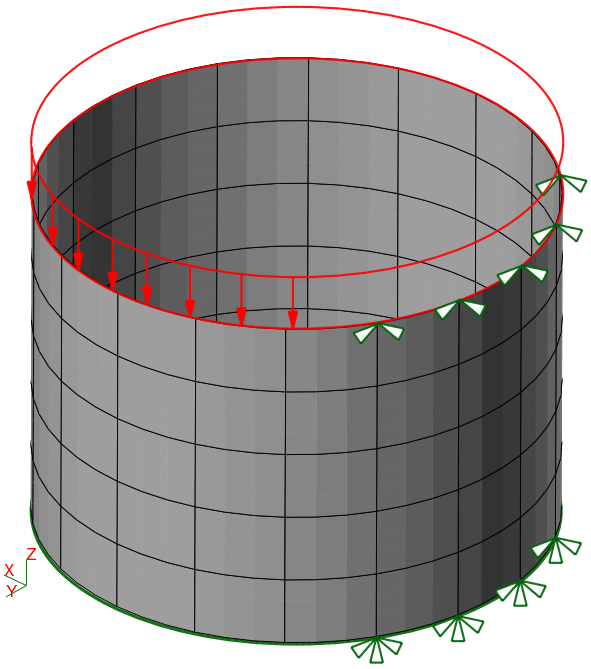

Figure 1: Finite element model of the cylindrical shell with schematic loads and supports.

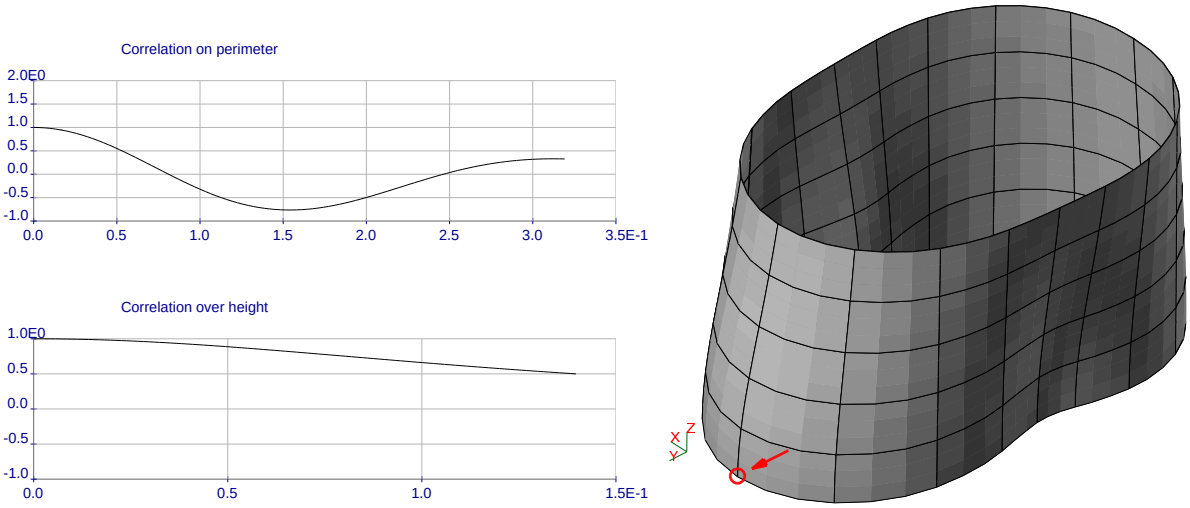

Figure 2: Correlation functions over perimeter (above) and height (below) and spatial correlation structure with respect to the marked node (right).

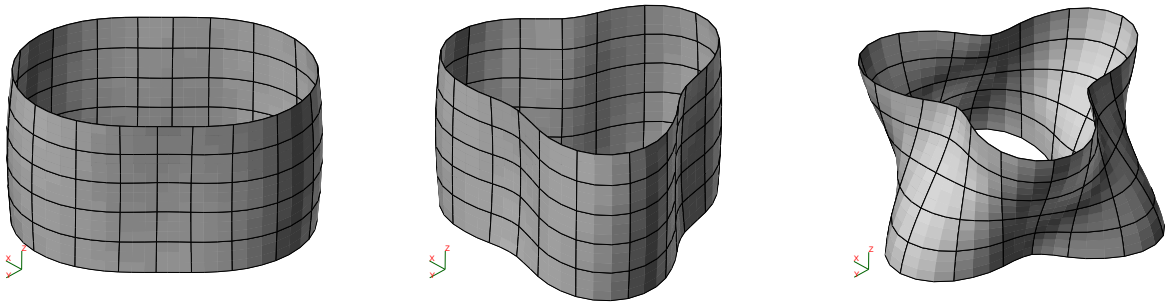

Figure 3: Selected eigenvectors of the covariance matrix.

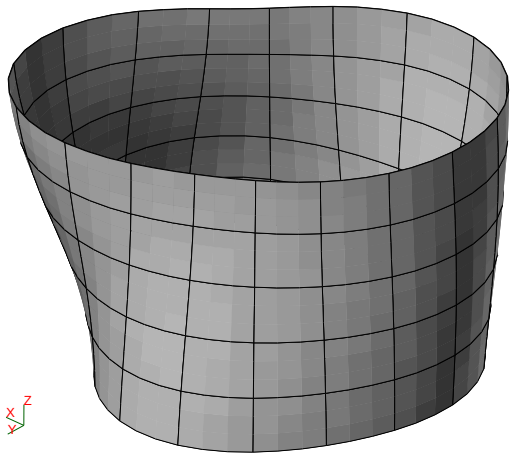

Figure 4: Sample of an imperfection shape (magnified).

The limit load for reliability analysis is adopted from the buckling load of the perfect structure subtracted a "safety margin". The limit state function reads

$$g(\mathbf{X}) = F_{buckling} - 34 \text{ kN} \leq 0 \tag{17}$$

Failure is defined as the event, that the limit state function takes values of less than zero. For an overview of methods to compute the probability of failure, see e.g. Bayer (1999).

5 Results

5.1 Preliminary studies

A robustness analysis is performed with optiSLang, wherein all 396 variables of the random field are involved and, among others, their relations to the critical buckling loads of the simulated imperfect structures are examined.

No significant linear correlations could be found. Instead, strong quadratic correlations are observed between the first 14 random variables (where "first" indicates those variables with the highest variances, i.e. highest eigenvalues after decomposition of the covariance matrix, cf. sect. 2.2) and the critical load. For variables of order higher than 14, the quadratic correlation coefficients are close to zero. The quadratic correlation matrix as explained in section 3 is displayed in fig. 5. The nonlinear dependency becomes obvious by an anthill (or scatter) plot, that is a plot of realizations of input and output variable pairs, see fig. 6 for an example. Fig. 7 shows a histogram of the sampled critical load and a probability density of Weibull type, which was the type of highest acceptance in the fit test. Since all input variables are of Gaussian type but the output is not, this is another evidence for the non–linear relation.

Based on the quadratic regression of eq. (13) for the buckling load, with each random variable set in for X successively, the coefficients of determination, eq. (14), are computed. The sum of all values is less than 100 %. That means, the variance of the critical load cannot be fully explained by a quadratic relation to the input variables. The results are sorted and plotted as bar diagram, fig. 8. The strongest influences can easily be identified. A closer look reveals that not all of the "first 14" variables (see above) are most relevant, but a few variables assigned to higher order eigenvectors of the covariance matrix as well. The eigenvalues and eigenvectors of the covariance matrix (used as random amplitudes and shape functions in the reliablity analysis) which are selected by the criterion of the "top 14" coefficients of determination are that of order 1, 2, 5, 6, 15, 21, 22, 26, 29, 30, 32, 34, 83 and 197.

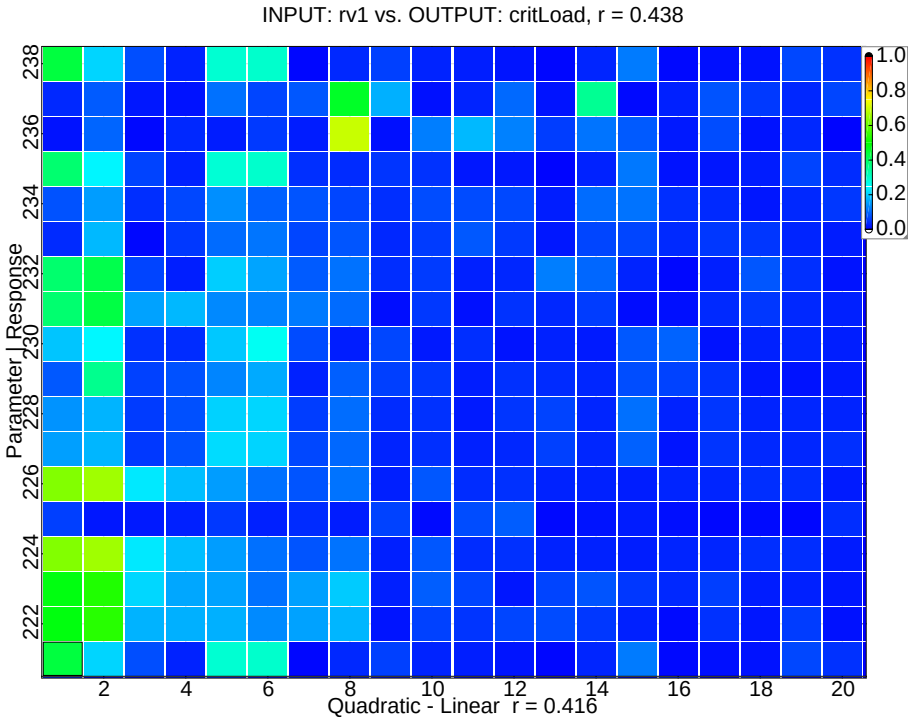

Figure 5: Matrix (part) of quadratic correlation coefficients. The lowest row shows correlations of the critical load with the first 20 random variables.

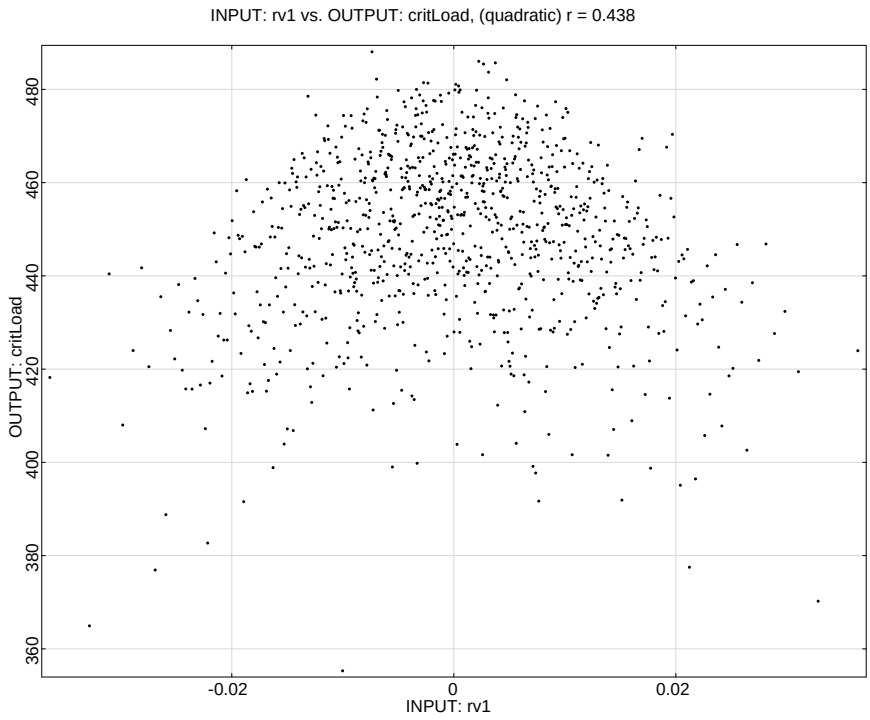

Figure 6: Anthill plot of critical load vs. first random variable.

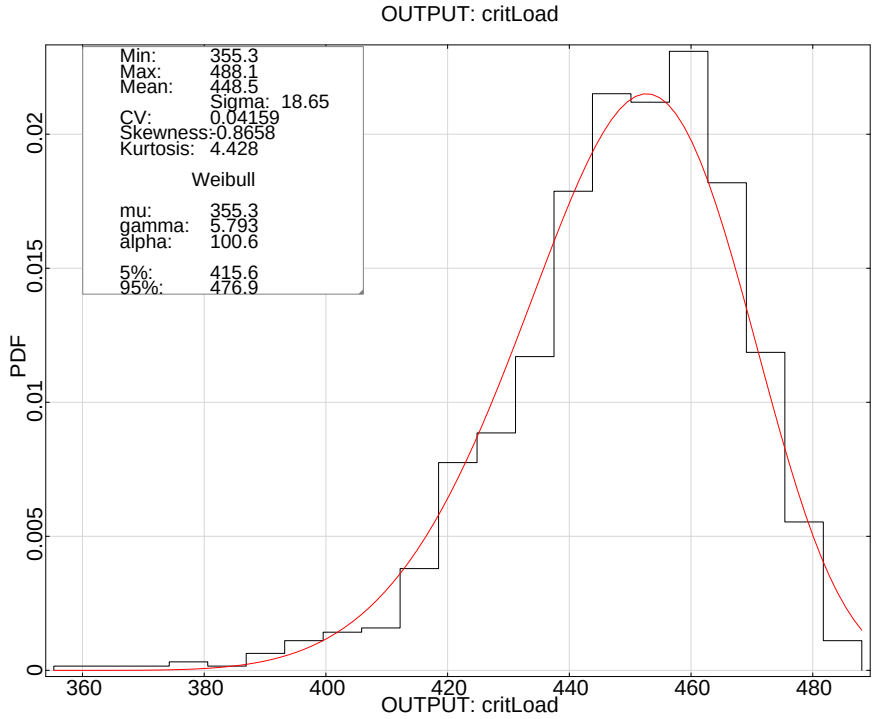

Figure 7: Histogram of the sampled critical load and fitted Weibull distribution.

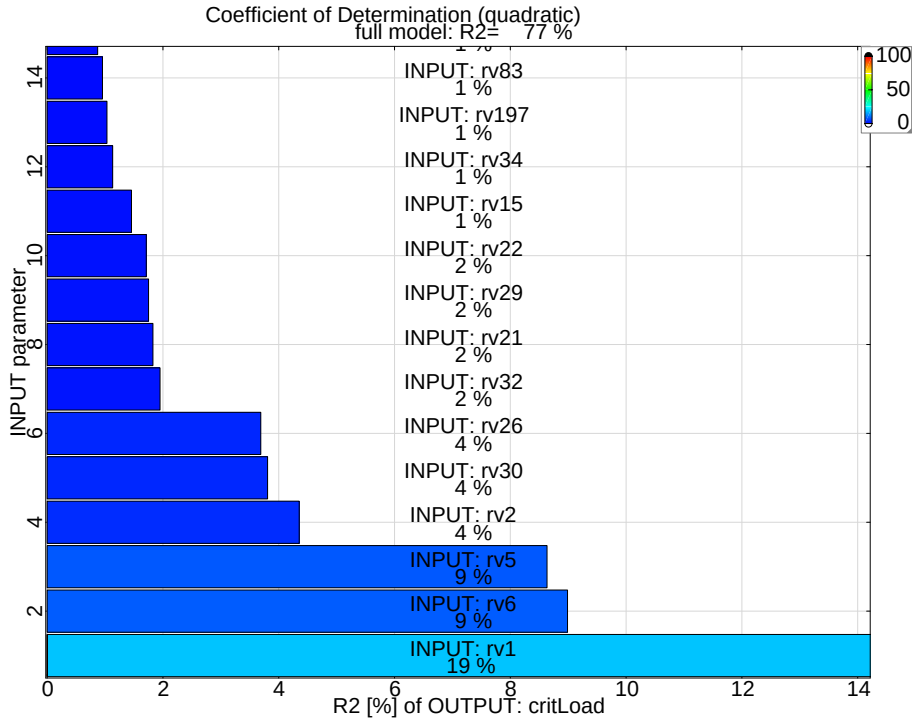

Figure 8: The top 14 coefficients of determination of critical load, quadratic regression model for random input variables.

5.2 Reliability Analysis

The reliability of the structure is studied by means of a Monte Carlo simulation (Bayer, 1999; Hammersley and Handscomb, 1967; Rubinstein, 1981), with the limit state function as defined by eq. (17). Three variants are computed: as a reference, the full representation of the random field, which employs 396 random variables, is used. Second, the "first 14" variables were selected by the criterion described on page 4 and third, a set of random variables with the "top 14" coefficients of determination, cf. sect. 5.1. In each case, a sample with 36000 realizations is generated by Latin Hypercube Sampling (McKay et al., 1979; Florian, 1992; Huntington and Lyrintzis, 1998). No other variance reduction scheme such as Importance Sampling is applied. Because the random field defines the structural geometry and hence the structural behaviour, the limit state function cannot be programmed explicitely, but a linear buckling analysis as explained in section 4.2 is carried out for each sample.

The failure probabilities computed with the different sets of variables are listed in table 2. Since Monte Carlo is a statistical method, the so–called statistical error is listed as well. This is the standard deviation of the estimator of the failure probability, which gives information about the confidence in the result. The simulation results with all variables and the "top 14" selection show a good quality. With the "first 14" set of variables, the probability of failure is underestimated by more than a magnitude. This set of random variables is able to represent the random field in good quality, but is not able to model the structural behaviour. The result obtained with the "top 14" selection is close to the reference, although it tends to be lower, too. Obviously, this selection criterion provides a good compromise for both modelling the stochastic and the mechanical problem.

Table 2: Probabilities of failure for different sets of random variables.

No. of random variables Selection criterion	396 none (all)	14 highest $\sigma^2_{X_i}$ ("first 14")	14 highest R^2 ("top 14")
Prob. of failure P_f	$9.7 \cdot 10^{-3}$	$2.8 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$
Statistical error σ_{P_f} cov(P_f)	$5.2 \cdot 10^{-4}$ 5 %	$8.8 \cdot 10^{-5}$ 32 %	$3.2 \cdot 10^{-4}$ 9 %

6 Conclusions

In this study, a procedure for calculating the reliability of a structural system with random imperfections was developed and tested. Such a complex problem still needs preliminary studies, but is already applicable to realistic problems. Within such preparations, attention must be paid on the analysis of the mechanical problem and choice of the calculation method. A compromise between accuracy and computing time should be found.

The reliability was computed by the Monte Carlo method, with the imperfections modelled as random fields. This may require a huge number of random variables, which in turn may cause numerical difficulties in the computation of the probability of failure. Hence another important task in the preparation phase is a suitable selection of the random variables. It is suggested to perform a robustness analysis for this purpose, which requires relatively few additional samples compared to the sample size needed for the reliability analysis itself. The robustness analysis comprises, among other functions, a quadratic regression of the state function by the input random variables. It turned out that for the underlying geometrically non–linear problem, this approach helps to detect the important input variables.

The kind of reliability analysis presented here is called non–parametric, because it depends on stochastic properties defined continuously on the entire structure, but not on geometry parameters. Quite often the randomness of the geometry is introduced by few random structural parameters, such as radius, material thickness etc. These values can be generated by CAD programs. In fact, only variants of the perfect structure are generated that way. It is impossible to consider the random fluctuations over space, which result e.g. from manufacturung tolerances. The proposed approach is more effective for this class of problems.

The efficiency of the reliability analysis still can be improved, e.g. by application of Response Surface methodology or variance reducing Monte Carlo techniques. This is part of current software developments which will become available in the future by integrating all necessary functions within optiSLang.

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