Robust Design Optimization

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Abstract

Within the robust design optimization, the statistical variability of the design parameter is considered. The most general method for solving optimization problems under uncertainties is the well established Monte Carlo simulation method. However, the major shortcoming of this approach is its vast need of computational resources (the number of solver runs required), and these cannot be presumed in general situations.

This paper reviews theories and methodologies that have been developed to solve optimization problems under uncertainties. In the first part the paper gives an overview over the state of the art in stochastic optimization methods such as robust design and reliability-based design optimization.

In addition, new adaptive response surface techniques as well as evolutionary algorithm in combination with first order reliability methods in robust design optimization and reliability-based optimization are developed. A numerical example from structural analysis under static and dynamic loading conditions shows the applicability of these concepts. The probabilistic and structural analysis tasks are performed with the optiSLang and SLang software packages.

Keywords: robust design optimization, reliability-based design optimization, design for six sigma, evolutionary algorithm, adaptive response surfaces, first order reliability methods

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1 Introduction

1.1 Challenges on virtual prototyping and multidisciplinary optimization

Methods of multidisciplinary optimization have obtained an increasing importance in the design of engineering problems for improving the design performance and reducing costs. The virtual prototyping is an interdisciplinary process. Such a multidisciplinary approach requires to run different solvers in parallel and to handle different types of constraints and objectives. Arbitrary engineering software and complex non-linear analyses have to be connected. Resulting optimization problems may become very noisy, very sensitive to design changes or ill-conditioned for mathematical function analysis (e.g. non-differentiable, non-convex, non-smooth).

During the last years, many challenges on virtual prototyping have occurred. Product life cycles are expected to last for as little as a few months, and more and more customized products are developed, e.g. 1700 car models compared to only 900 ten years ago. The engineer's focus is more and more on "built-in-quality" and "built-in-reliability". The products are developed in the shortest amount of time, and, inspire of that, they have to be safe, reliable and robust. Some markets require optimized product designs to be robust, e.g. defense, aerospace, jet engine, nuclear power, biomedical, oil industry and other mission critical tasks.

At the same time, the numerical models become increasingly detailed and numerical procedures become more and more complex. Substantially more precise data are required for the numerical analysis. Commonly, these data are random parameters. From this it follows that the optimization process includes uncertainties or stochastic scatter of design variables, objective function and restrictions as shown in Figure 1. Furthermore, the optimized designs lead to high imperfection sensitivities and tend to loose robustness. Using a multidisciplinary optimization method, the deterministic optimum design is frequently pushed to the design space boundary. The design properties have no room for tolerances or uncertainties. So the assessment of structural robustness, reliability and safety will be more and more important. Because of that, an integration of optimization and stochastic structural analysis methods is necessary.

1.2 Design for Six Sigma

Six Sigma is a quality improvement process to optimize the manufacturing process in a way that it automatically produces parts conforming to the Six Sigma quality level, as shown in Figure 1.1. Motorola documented more than $16 Billion in savings as a result of their Six Sigma efforts[1]. Since then, hundreds of companies around the world have adopted Six Sigma as a way of doing business.

In contrast, Design for Six Sigma optimizes the design itself such that the part conforms to Six Sigma quality even with variations in manufacturing, as shown in Figure 3. Design for Six Sigma is a concept to optimize the design such that the parts conform with Six Sigma quality, i.e. quality and reliability are explicit optimization goals. Robust design is often synonymous to "Design for Six Sigma" or "reliability-based optimization". The possible sigma levels start at 1,2 σ (robust design optimization) and go up to 6 σ
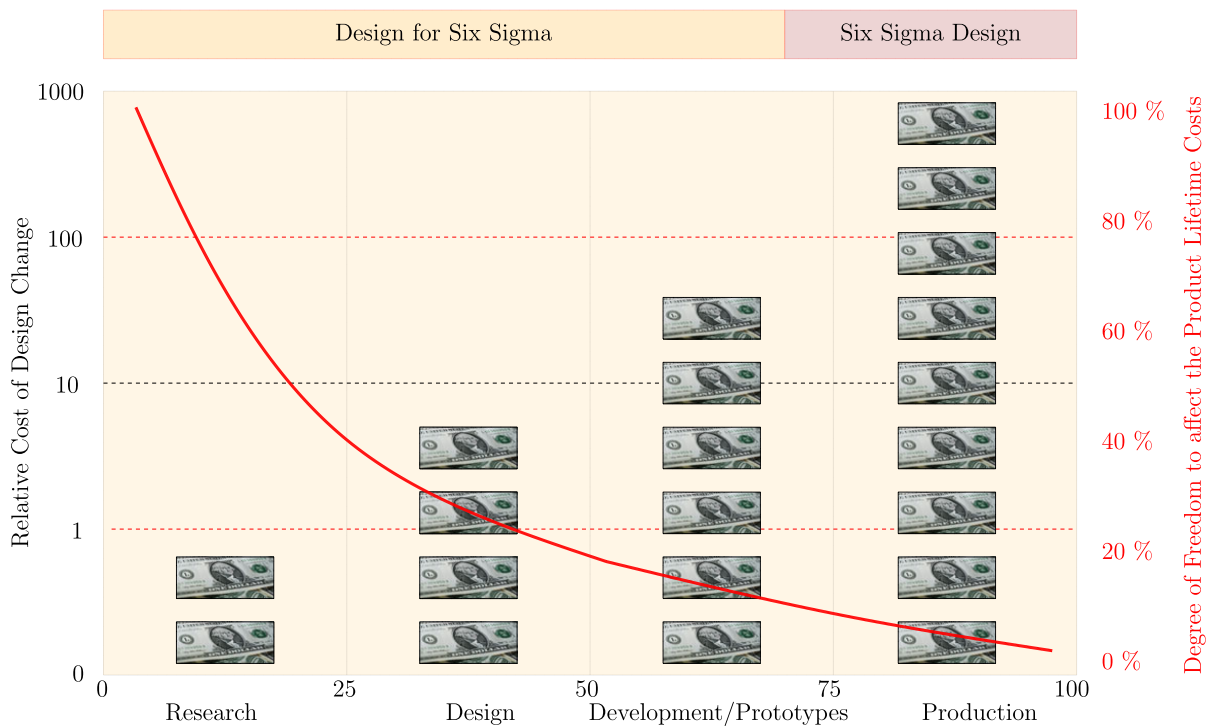
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[1]source: www.isixsigma.com/library/contentc020729a.asp
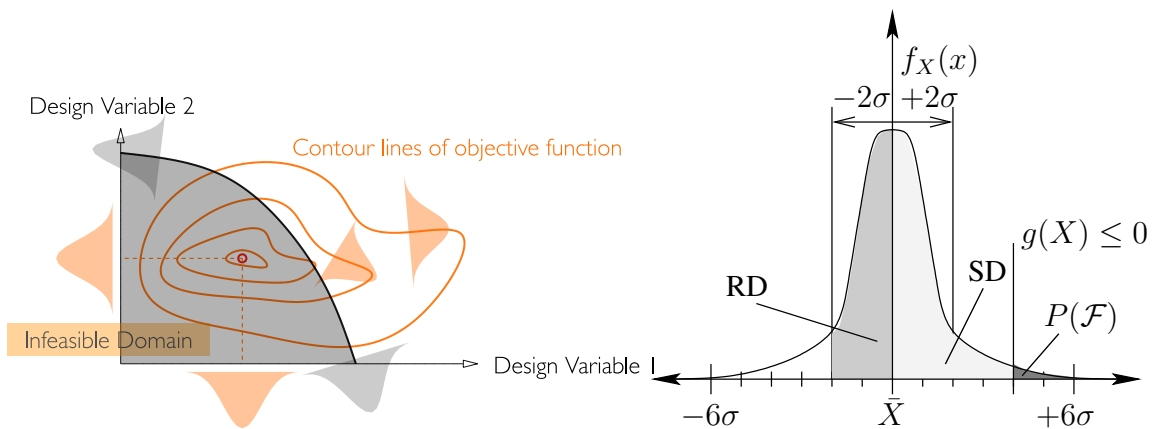
Figure 1: Sources of uncertainty in design optimization.

Figure 2: Normal distribution $f_X(x)$ with lower and upper specification limit on 2σ and 6σ level. Robust design (RD) and safety design (SD) ($\geq \pm 2\sigma$) depending on chosen limit state function $g(X) \leq 0$, e.g. stress limit state.

Figure 3: Product Development Phases: within the "Design for Six Sigma" the degree of freedom to affect the product lifetime cost is very high and the cost of design change is propositional moderate in contrast to "Six Sigma Design" concept to optimize the manufacturing processes only.

Sigma level	Percent variation	Probability of failure $P(\mathcal{F})$	Defects per million (short term)	Defects per million (long term)
$\pm 1\sigma$	68.26	$3.17 \cdot 10^{-1}$	317400	697700
$\pm 2\sigma$	95.46	$4.54 \cdot 10^{-2}$	45400	308733
$\pm 3\sigma$	99.73	$2.7 \cdot 10^{-3}$	2700	66803
$\pm 4\sigma$	99.9937	$6.3 \cdot 10^{-5}$	63	6200
$\pm 5\sigma$	99.999943	$5.7 \cdot 10^{-7}$	0.57	233
$\pm 6\sigma$	99.9999998	$2.0 \cdot 10^{-9}$	0.002	3.4

Table 1: Sigma level depending on the variation of the normal distribution, defects per million and associated probability of failure $P(\mathcal{F})$. A probability of 3.4 out of 1 million is achieved when the performance target is 4.5 σ away from the mean value (short term). The additional 1.5 σ (long term) leading to a total of 6 standard deviations are used as a safety margin to allow for "drift of the mean value" in the properties and environment which the product can see over its lifetime.

(reliability-based design optimization) (Koch et al. (2004)), as shown in Table 1. Within the robust design optimization, the statistical variability of the design parameter is considered. The most general method for solving robust design optimization problems is the well established Monte Carlo simulation method. However, the major shortcoming of this approach is its vast need of computational resources (the number of solver runs required), and these cannot be presumed in general situations.

1.3   Robust design optimization

Optimized designs within the sigma level $\leq \pm 2\sigma$ are characterized as robust design (RD). The objective of the robust design optimization (e.g. Hwang et al. (2001); Ben-Tal and Nemirovski (2002); Doltsinis and Kang (2004)) is to find a design with a minimal variance of the scattering model responses around the mean values of the design parameters (see Byrne and Taguchi (1987); Phadke (1989)).

Other approaches for an evaluation of the design robustness, e.g. the linear approximation of "scattering" solver responses (see e.g. Abspoel et al. (1996)) or the variance estimation in genetic programming (see e.g. Pictet et al. (1996); Branke (1998)), independently of given parameter distributions will not be subject of the following remarks as they are not to be counted to robust design optimization methods in a stricter sense.

1.4   Reliability-based optimization

In the reliability-based optimization, the optimization problem can be enhanced by additional stochastic restrictions ensuring that prescribed probabilities of failure can not be exceeded. Furthermore, the probability of failure itself can be integrated into the objective function. Frequently, the search for the optimum by means of deterministic optimization is combined with the calculation of the failure probability, e.g. using the first- order second-moment analysis (FOSM) (e.g. Melchers (2001)). A more promising combination may under certain circumstances involve the first and second order reliability methods (FORM/SORM) (e.g. Choi et al. (2001a); Allen et al. (2004); Allen and Maute (2004)).

Within the deterministic optimization, a calculation of the failure probability of individual designs has to be performed in order to be able to properly evaluate these designs. Therefore, special attention has to be paid to the cost efficiency of this calculation. As an example, for smooth and well-scaled objective functions with few continuous design parameters, the deterministic optimization as well as the determination of the failure probability that is included within the optimization iteration loop may be performed by means of gradient based programming (e.g. Sequential Quadratic Programming, see Schittkowski (1985)).

In Kharmanda et al. (2002) a decrease of the numerical expense of these two nested iterations is attempted by substituting the deterministic objective function as well as the limit state function on which the point of largest probability density is searched within FORM by a single objective function in a hybrid design space. However, this leads to an enlargement of the design space for the gradient based programming.

In the reliability-based optimization, frequently approximation function are applied that at the same time approximate the design space and the space of random parameters by means of a meta-model, e.g. in Choi et al. (2001b); Youn et al. (2004); Yang and Gu (2004); Rais-Rohani and Singh (2004). Successful industrial applications of these methods can amongst others be found in Youn and Choi (2004).

In Royset and Polak (2004), a linear approximation of the limit state function serves as a constraint of the optimization problem. An improvement of the optimization result is tempted in Royset et al. (2003) by taking into account the gradients of the limit state function.

However, in the robust optimization (see Chen et al. (2004); Wilson et al. (2001)) as well, different approximation models in combination with an appropriate variance determination are used, e.g. global polynomial approximations and Kriging models. Their use is restricted to problems with few random variables and few optimization variables ($n \leq 5$).

# 2 Robust design optimization

2.1 Introduction

Reliability-based design optimization   In reliability-based design optimization, the deterministic optimization problem

$$
\begin{aligned}
f(d_1, d_2, \ldots d_{n_d}) &\rightarrow \min \\
g_k(d_1, d_2, \ldots d_{n_d}) &= 0; \ k = 1, m_e \\
h_l(d_1, d_2, \ldots d_{n_d}) &\geq 0; \ l = 1, m_u \\
d_i &\in [d_l, d_u] \subset \mathbb{R}^{n_d} \\
d_l \leq d_i &\leq d_u \\
d_i &= E[X_i]
\end{aligned}
\tag{1}
$$

with n_r random parameters $\mathbf{X}$ and n_d means of the design parameters $\mathbf{d} = E[\mathbf{X}]$ is enhanced by additional m_g random restrictions

$$
\int_{g_j(\mathbf{x}) \leq 0} \overset{n_r}{\cdots} \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} - P(\mathbf{X} : g_j(\mathbf{X}) \leq 0) \leq 0; \ j = 1, m_g
\tag{2}
$$
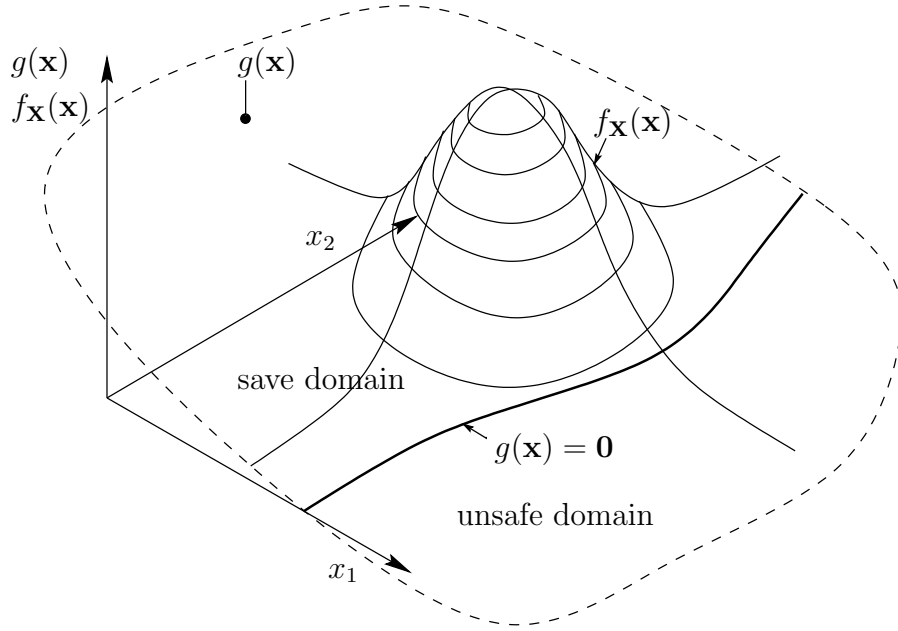
Figure 4: The state function $g(\mathbf{x})$ of a numerical model is given implicitly, e.g. is result of a finite element analysis depending on several design responses. The failure condition leads to a unknown deterministic limit state function $g(\mathbf{x}) = 0$, where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function.

with the joint probability density function of the basic random variables $f_{\mathbf{X}}(\mathbf{x})$ and m_g limit state functions $g_j(\mathbf{x}) \leq 0$ (see Figure 2.1). The probability of failure in (2) is calculated applying the reliability analysis.

Furthermore the objective function can be enhanced by additional criteria such as minimization of the probability of failure $P(\mathcal{F})$

$$f(d_1, d_2, \ldots d_{n_d}, P(\mathcal{F})) \to \min \tag{3}$$

with

$$P(\mathcal{F}) = \int \overset{n_r}{\cdots} \int_{g_j(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{4}$$

Robust design optimization Within the robust design optimization, the objective (1) is enhanced by the requirement to minimize the variances $\sigma^2_{X_i}$
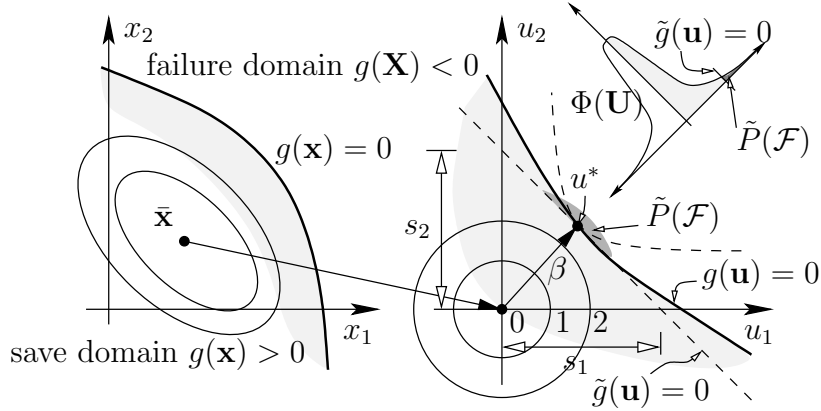
$$f(d_1, d_2, \ldots d_{n_d}, \sigma^2_{X_1}, \sigma^2_{X_2}, \ldots \sigma^2_{X_{n_r}}) \to \min \tag{5}$$

with

$$\sigma^2_{X_i} = \frac{1}{M-1} \sum_{k=1}^{M} \left(x_i^k - \mu_{X_i} \right)^2$$

2.2 Evolutionary algorithm in combination with the first-order reliability method (FORM)

Evolutionary algorithm For a multitude of optimization problems in structural mechanics, the precision of the input data must be doubted. The deviations from the target

Figure 5: Transformations from correlated non-Gaussian variables $\mathbf{X}$ to uncorrelated Gaussian variables $\mathbf{U}$ with zero mean and unit variance. Gradient-based optimization for design point search (most probable point β-point). Approximation of the limit state function $g(\mathbf{u})$ by a linear function $\tilde{g}(\mathbf{u})$ in standard Gaussian space.

values or nominal values can often be reasonably described by random variables. Especially such problems without any overlapping of design variables and random variables permit to choose completely different strategies for optimization and stochastic analysis in order to ably exploit the their advantages.

Evolutionary algorithm is reasonably used in cases when the objective function's and or the restrictions' dependency on the design parameters is not differentiable or not even continuous. Evolutionary algorithms are stochastic search methods that mimic processes of natural biological evolution like adaption, selection and variation. Based on the principle 'survival of the fittest' a population of artificial individuals search the design space of possible solutions in order to find a better approximation for the solution of the optimization problem.

Many variants have been implemented over the past decades, based on the three main classes *genetic algorithms* (GA) developed by Holland (1975) and Goldberg (1989), *evolution strategies* (ES) introduced by Rechenberg (1973) and Schwefel (1981) and *evolutionary programming* (EP) developed by Fogel et al. (1966). These algorithms have been originally developed to solve optimization problems where no gradient information is available, like binary or discrete search spaces, although they can also be applied to problems with continuous decision variables. optiSLang (Bucher et al. (2001)) provides two different EA implementations: a real-coded genetic algorithm and a customizable evolutionary algorithm framework whose different features.

FORM - First Order Reliability Method Typically, the failure probabilities of well designed systems are small. Therefore, a reliability method has to be applied that provides these value at a reasonable expense. This can be with good success the first-order reliability method (FORM) (Hasofer and Lind (1974); Rackwitz and Fießler (1978); Shinozuka (1983); Hohenbichler and Rackwitz (1988); Tvedt (1983); Breitung (1991)) for problems the restrictions of which (usually including the failure probability in some form) are depending on the stochastic variables in a differentiable way.

The FORM-Concept is based on a description of the reliability problem in standard Gaussian space. Hence transformations from correlated non-Gaussian variables $\mathbf{X}$ to

uncorrelated Gaussian variables $\mathbf{U}$ with zero mean and unit variance are required. This step is called Rosenblatt-transformation.

In order to determine the failure probability $P(\mathcal{F})$ in Eq. (2) the limit state function $g(\mathbf{u})$ can be approximated by a Taylor expansion $\bar{g}(\mathbf{u})$. The optimal expansion point is the design point with coordinates $(s1, .., s_N)$, which is the point on $g(\mathbf{u})$ closest to the origin in Gaussian space, as shown in Fig. 2.2. From a safety engineering point of view, the point $\mathbf{x}^*$ corresponding to $\mathbf{u}^*$ is called *most probable failure point* or *design point*. The distance β to the origin is called the reliability index. From the geometrical interpretation of the expansion point $\mathbf{u}^*$ in standard Gaussian space it becomes quite clear that the calculation of the design point can be reduced to an optimization problem:

$$\mathbf{u}^* : \mathbf{u}^T\mathbf{u} \rightarrow \text{Min.}; \quad \text{subject to: } g[\mathbf{x}(\mathbf{u})] = 0 \tag{6}$$

This leads to the Lagrange-function

$$L = \mathbf{u}^T\mathbf{u} + \lambda g(\mathbf{u}) \rightarrow \text{Min.} \tag{7}$$

Standard optimization procedures can be utilized to solve for the location of $\mathbf{u}^*$. In optiSLang (Bucher et al. (2001)) the NLPQLP algorithm is used, which is based on a sequential quadratic programming (SQP) method. Within this method the gradients of the objective function and the restrictions need to be determined, which is performed using finite differences. For further details see of the optimization method see Schittkowski (2004).

In the next step, the exact limit state function $g(\mathbf{u})$ is replaced by a linear approximation $\bar{g}(\mathbf{u})$. From this, the probability of failure is easily determined to be

$$P(\mathcal{F}) = P\left[Z \leq 0\right] \approx P\left[\bar{Z} \leq 0\right] \tag{8}$$

$$= \int_{-\infty}^{0} f_{\bar{Z}}(\bar{z})d\bar{z} = F_{\bar{Z}}(0) \tag{9}$$

$$= \Phi\left(\frac{-E\left[\bar{Z}\right]}{\sigma_{\bar{Z}}}\right) = \Phi\left(-\frac{1}{\frac{1}{\beta}}\right) \tag{10}$$

$$= \Phi\left(-\beta\right) \tag{11}$$

This result is exact, if $g(\mathbf{u})$ is actually linear. In general FORM gives a good approximation of the failure probability.

In this context it has to be considered that genetic algorithms generally implement the restrictions in the form of penalty terms. Thus, the choice of an appropriate penalty method is of a certain importance.

2.3 Adaptive response surfaces

2.3.1 Introduction

The response surface methodology (RSM) is one of the most popular strategies for nonlinear optimization. Due to the inherent complexity of many engineering optimization problems it is quite alluring to approximate the problem and to solve the optimization in a smooth sub-domain by applying response surface methodology.

Usually, for a large number of real-life design optimization problems the objectives and constraints are determined as a result of expensive numerical computations. Furthermore, the function values and their derivatives may contain numerical noise and the calculability of some of the response functions is domain-dependent, e.g. situations when these functions cannot be evaluated at some points of the design space. Especially, to solve this kind of optimization the *adaptive response surface methodology* (ARSM) (see e.g. Etman et al. (1996); Toropov and Alvarez (1998); Abspoel et al. (1996); Stander and Craig (2002); Kurtaran et al. (2002)) is developed as a consequent combination of optimization strategy and response surface methodology.

Of course the accuracy of the approximation compared to the real problem has to be checked and verified. Mainly three factors influence the accuracy of a response surface:

1. The number and distribution of support points. The systematic sampling schemes try to place the support points in an optimized way according to the boundary of the design space and the distance of the support points. For reasonably smooth problems, the accuracy of response surface approximations improves as the number of points increases. However, this effect decreases with the degree of oversampling.

2. The choice of the approximation function. In general, higher order functions are more accurate. Linear functions require fewest support points, but are weak approximations. Quadratic functions are most popular. The second order polynom results in a smooth approximation function and is well scaled for gradient based optimizers. Using polynomials higher than second order may only result in higher local accuracy with many sub-optima.

3. The design space. The overall possible design space is given by the lower and upper boundaries of the optimization parameters. Of course the smaller the approximated subregions, the greater the accuracy. In practical problems we will start with the overall design space and further investigate smaller subregions.

In contrast to the RSM the ARSM uses a subregion of the global parameter range to approximate the responses. Starting with a presumably large subregion the iteration moves and shrinks the subspace till a solution converges to an optimum. This strategy will be denoted as *move limit strategy*. Usually, this is done using low level trial functions (e.g. linear and quadratic polynomial functions).

In Roos et al. (2006) a new adaptive response surface method on random space is introduced in combination with an advanced moving least square approximation. Additionally, on the design space a modified moving least square function is introduced to approximate the responses with high accuracy and efficiency using of all calculated support points.

2.3.2 Moving Least Square Approximation

Moving least square (MLS) functions can approximate locally clustered support point samples with higher local approximation quality. In addition MLS improve the response surface model using additional support points. MLS is formulated as

$$\hat{y}(\mathbf{x}) = \sum_{i=1}^{n_b} h_i(\mathbf{x}) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x})\, \mathbf{a}(\mathbf{x}) \tag{12}$$
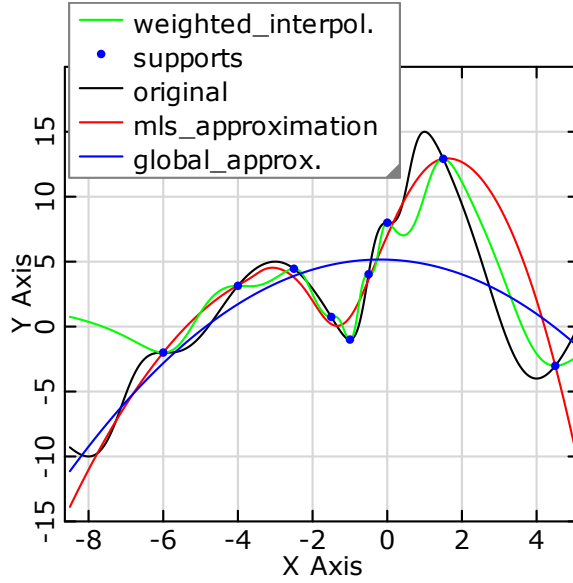
Figure 6: A comparison of different approximation functions: global 2nd order polynomial, moving least square with quadratic basis and constant basis (weighted interpolation).

with a predefined number of basis terms n_b, a vector of basis functions $\mathbf{h}$ and the associated vector of the coefficients $\mathbf{a}$. Lancaster and Salkauskas (1986) formulates a local MLS approximation as

$$\hat{y}(\mathbf{x}, \mathbf{x}_j) = \sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}_j)\, \mathbf{a}(\mathbf{x})$$

with $j = 1, ..., n_s$ support points. The approximate coefficient vector $\mathbf{a}$ can be calculated using the weighted least square postulate

$$
\begin{aligned}
S(\mathbf{x}) &= \sum_{j=1}^{n_s} w(\mathbf{x} - \mathbf{x}_j) \left(\hat{g}(\mathbf{x}, \mathbf{x}_j) - \hat{g}(\mathbf{x}_j) \right)^2 \\
&= \sum_{j=1}^{n_s} w(\mathbf{x} - \mathbf{x}_j) \left(\sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) - \hat{g}(\mathbf{x}_j) \right)^2 \\
&= (\mathbf{Ha} - \mathbf{g})^T \mathbf{W}(\mathbf{x})(\mathbf{Ha} - \mathbf{g}) \rightarrow \min
\end{aligned}
\tag{13}
$$

with the weighting function $w(\mathbf{x} - \mathbf{x}_j)$ and

$$
\begin{aligned}
\mathbf{g} &= \begin{bmatrix} y(\mathbf{x}_1) & y(\mathbf{x}_2) & ... & y(\mathbf{x}_{n_s}) \end{bmatrix}^T \\
\mathbf{H} &= \begin{bmatrix} \mathbf{h}(\mathbf{x}_1) & \mathbf{h}(\mathbf{x}_2) & ... & \mathbf{h}(\mathbf{x}_{n_s}) \end{bmatrix}^T \\
\mathbf{W}(\mathbf{x}) &= \text{diag}\begin{bmatrix} w(\mathbf{x} - \mathbf{x}_1) & w(\mathbf{x} - \mathbf{x}_2) & ... & w(\mathbf{x} - \mathbf{x}_{n_s}) \end{bmatrix}
\end{aligned}
$$

The least square error $S(\mathbf{x})$ may be a minimum in case that the partial gradients

$$\frac{\partial S(\mathbf{x})}{\partial \mathbf{a}} = 0$$

are zero. So using the equation (13) a linear equation system gives an estimation of the coefficient vector $\mathbf{a}$

$$\mathbf{a(x)} = \mathbf{M}^{-1}(\mathbf{x}) \ \mathbf{B(x)} \ \mathbf{g} \tag{14}$$

with

$$\mathbf{M(x)} = \mathbf{H}^T \ \mathbf{W(x)} \ \mathbf{H}$$
$$\mathbf{B(x)} = \mathbf{H}^T \ \mathbf{W(x)}$$

Cause the matrix of the basis function $\mathbf{M(x)}$ should be non-singular always a sufficient number of n_s immediate neighbor support points have to be available. The number must be at least as large as number of the basis terms. The equation (14) inserted in (12) gives the approximation function

$$\hat{y}(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}) \ \mathbf{M}^{-1}(\mathbf{x}) \ \mathbf{B(x)} \ \mathbf{g}$$

An accurate as possible approximation quality requires a weighting function which is larger than zero $w(\mathbf{x} - \mathbf{x}_j) > 0$ and monotonically decreasing $w(\|\mathbf{x} - \mathbf{x}_j\|)$ inside of a small sub space $\Omega_s \subset \Omega$. So the influence of supports far from the actual coordinates is unimportant. A uniform weighting is given by a symmetry condition $w(\mathbf{x} - \mathbf{x}_j) = w(\|\mathbf{x} - \mathbf{x}_j\|)$. Usually, an exponential function is used in this way:

$$w\left(\frac{\|\mathbf{x} - \mathbf{x}_j\|}{D}\right) = \begin{cases} e^{-\left(\frac{\|\mathbf{x} - \mathbf{x}_j\|}{D\alpha}\right)^2} & \|\mathbf{x} - \mathbf{x}_j\| \le 1 \\ 0 & \|\mathbf{x} - \mathbf{x}_j\| > 1 \end{cases}$$

with a constant

$$\alpha = \frac{1}{\sqrt{-\log 0.001}}$$

and a influence radius D to choose. It is obvious that the smaller D the better the response values of the support points fit the given values. But as mentioned above at least n_b support points have to be available in every point to be approximated. Therefore it is possible that a D has to be chosen which leads to a large shape function error at the support points. To avoid these problems a new regularized weighting function was introduced by Most and Bucher (2005):

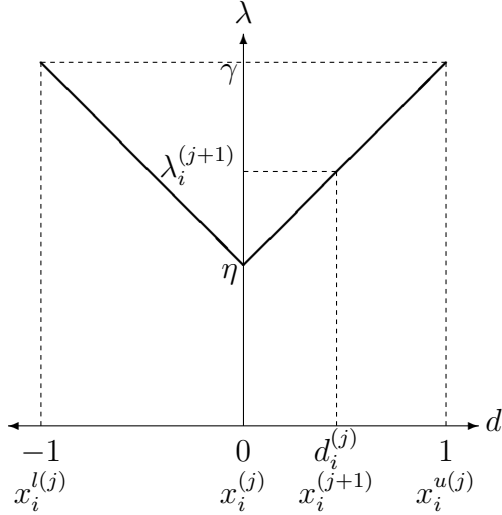$$w_R(\|\mathbf{x} - \mathbf{x}_j\|) = \begin{cases} \dfrac{\hat{w}_R(\|\mathbf{x} - \mathbf{x}_j\|)}{\sum\limits_{i=1}^{n_s} \hat{w}_R(\|\mathbf{x} - \mathbf{x}_i\|)} & \|\mathbf{x} - \mathbf{x}_j\| \le D \\ 0 & \|\mathbf{x} - \mathbf{x}_j\| > D \end{cases} \tag{15}$$
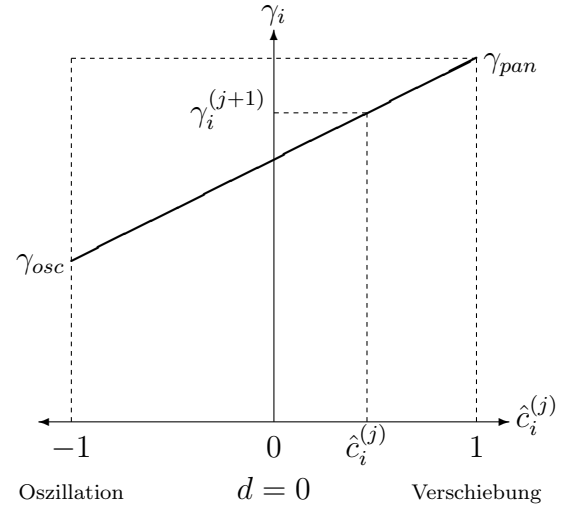
with

$$\hat{w}_R(d) = \frac{\left(\left(\frac{d}{D}\right)^2 + \varepsilon\right)^{-2} - (1 + \varepsilon)^{-2}}{(\varepsilon)^{-2} - (1 + \varepsilon)^{-2}} \ ; \ \varepsilon \ll 1 \tag{16}$$

It is recommended by the authors to use the value

$$\varepsilon = 10^{-5}$$

Figure 7: Contraction rate in relation to the movement indicator.

Figure 8: Contraction parameter in relation to the normalized oscillation indicator.

This new regularized weighting function works better than the exponential function. But if the ratio of the minimal distance among the supports to the extent of areas where are no supports becomes worse the same problems occur again. As a matter of fact a large D is needed to approximate for coordinates where no support points are around and a small D is needed for coordinates where are a lot of support points in order to reach a minimal approximation error. To comply with this conditions it is necessary to use a function $d(\mathbf{x})$ for the influence radius instead of a constant D. In comparison with the global 2nd order polynomial and MLS with constant basis (weighted interpolation) the Figure 6 shows the high accuracy and efficiency of the surrogate model using moving least square with quadratic basis and an adaptive influence radius for D.

### 2.3.3 Adaptive design of experiment

In conformity with the gradient based optimization, a start point must be determined. Typically, the start design is the center point $\mathbf{x}^{(0)}$ of the start region. The parameter boundaries depend on the center point and the global range $r_i^{(0)}$

$$x_i^{l(0)} = x_i^{(0)} - \gamma_{start}\ r_i^{(0)} \quad \text{and} \quad x_i^{u(0)} = x_i^{(0)} + \gamma_{start}\ r_i^{(0)} \quad i = 1, \ldots, n \qquad (17)$$

for all n parameters. The factor γ_{start} gives the relationship between global range and sub range. The default value γ_{start} is 50% ($\gamma_{start} = 0.5$) of the global range. If the subregion is outside the global range the subregion will be moved back inside the global boundaries.

The generation of support points depends on the order of the approximation function. Usually KOSHAL or D-optimal designs are used because they require the least number of support points. This means only as much (expensive) numerical calculations of the problem as necessary.

When the response values for all support points are calculated, the response surfaces for every single response value will be separately approximated. For these local approximations polynomials of first or second order are used (linear or quadratic approximation).
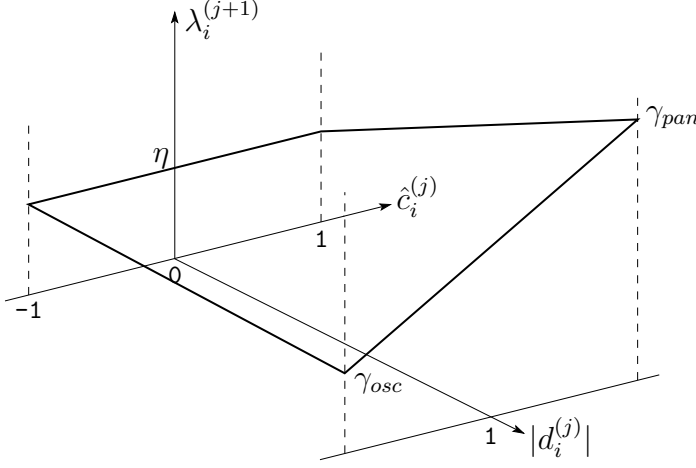
Figure 9: Determination function of λ_i depending on the movement and oscillation indicators, as well as on the three parameters γ_{pan}, γ_{osc} and η.
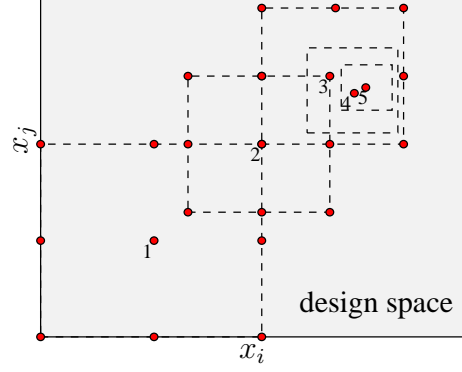
Figure 10: Sequential modification of the subregion.

In order to approximate all support points the moving least square function increases approximation accuracy.

If the next iteration will exceed these boundaries the algorithm will be stopped, as shown in Figure 10. Furthermore, the convergence of parameter and objective values will be analysed using a percentual limit for the changes from the previous iteration:

$$\left| \frac{x_i - x_{i-1}}{x_i} \right| * 100\% < \Delta x \tag{18}$$

The new subregion boundaries depend on the last optimum and the range of the appropriate subspace. A modification of the subspace can be classified in a pure panning, pure zooming or a combination of both (see Stander and Craig (2002)). Beginning with the start design every ascertained optimum is the new start and center point of the following subspace optimization. The displacement

$$\Delta x_i^{(j)} = x_i^{(j)} - x_i^{(j-1)} \tag{19}$$

describes the quantity of the subspace movement. The *movement indicator*

$$d_i^{(j)} = \frac{2 \Delta x_i^{(j)}}{r_i^{(j)}} \quad with \quad d_i^{(j)} \in [-1; 1]. \tag{20}$$

describes the quality of the displacement. A movement indicator equal to zero ($d_i^{(j)} = 0$) moves the range of the subregion only. If the found optimum is between the center point and the subspace boundaries $0 < |d_i^{(j)}| < 1$ then the subspace boundaries are modified and the subregion will be moved. The third case $|d_i^{(j)}| = 1$ leads to a subspace shift because the optimum is located on the boundaries of the subregion. The algebraic sign $sign(d_i^{(j)})$ indicates the relative movement direction of the subregion.

The contraction rate $\lambda_i^{(j)}$ describes the proportion of the last subregion range $r_i^{(j-1)}$ to the current range $r_i^{(j)}$. This rate is the relevant numeric determination of the new

subspace boundaries.

$$r_i^{(j)} = \lambda_i^{(j)} r_i^{(j-1)} \qquad \text{with} \qquad i = 1, \ldots, p \,;\; j = 1, \ldots, \text{n iterations}$$
$$x_i^{l(j)} = x_i^{(j)} - \frac{1}{2} r_i^{(j)} \qquad \text{(lower boundaries)} \tag{21}$$
$$x_i^{u(j)} = x_i^{(j)} + \frac{1}{2} r_i^{(j)} \qquad \text{(upper boundaries)}$$

For the determination of $\lambda_i^{(j)}$ a linear absolute value function (see Fig. 7)

$$\lambda_i^{(j)} = \eta + |d_i^{(j)}| \, (\gamma - \eta) \ . \tag{22}$$

is used. The *zoom parameter* η and the *contraction parameter* γ determine the threshold values of the linear function. The default values are $\eta = 0.5$ and $\gamma = 1.0$ (Kok and Stander (1999)) and $\eta = 0.5$ and $\gamma = 1.0$ (Kurtaran et al. (2002)) using quadratic function.

In order to avoid an alternating subregion between two states an oscillation criterion which influences the contraction parameter γ is needed. For this purpose an *oscillation indicator*

$$c_i^{(j)} = d_i^{(j)} d_i^{(j-1)} \tag{23}$$

is introduced or a *normalized oscillation indicator*

$$\hat{c}_i^{(j)} = \sqrt{|c_i^{(j)}|} \, sign(c_i^{(j)}) \tag{24}$$

respectively.

Typically, the parameter γ_{osc} is chosen between 0.5 and 0.7 and represents the shrinkage of the oscillation. The size of γ is determined by means of the parameters *panning parameter* γ_{pan} which represents the pure panning case with the last optimum on the subregion boundaries and the *oscillation parameter* γ_{osc} which shrinks the subspace range. Figure 8 shows the linear relationship

$$\gamma_i^{(j+1)} = \frac{1}{2} \left[\gamma_{pan} \left(1 + \hat{c}_i^{(j)} \right) + \gamma_{osc} \left(1 - \hat{c}_i^{(j)} \right) \right] \tag{25}$$

between contraction parameter and normalized oscillation indicator.

Consequently, the determination function of λ_i is a plain function which depends on the movement and oscillation indicators, as well as on the three parameters η, γ_{pan} and γ_{osc} which determine the magnitude of the function, as shown in Figure 9.

3 Example – Robust design optimization of a dynamic structure

3.1 Structural system

The aim of the classical optimization problem for structural elements is to minimize the mass while observing deformation or strength restrictions. As an example for a robust design optimization the mass of this simple beam with rectangular cross section (d, h)
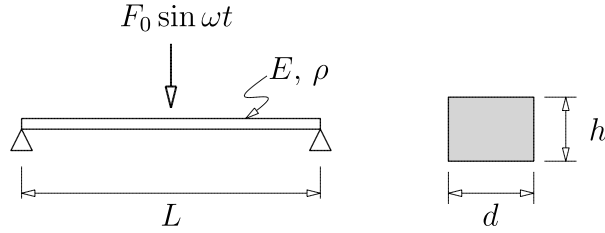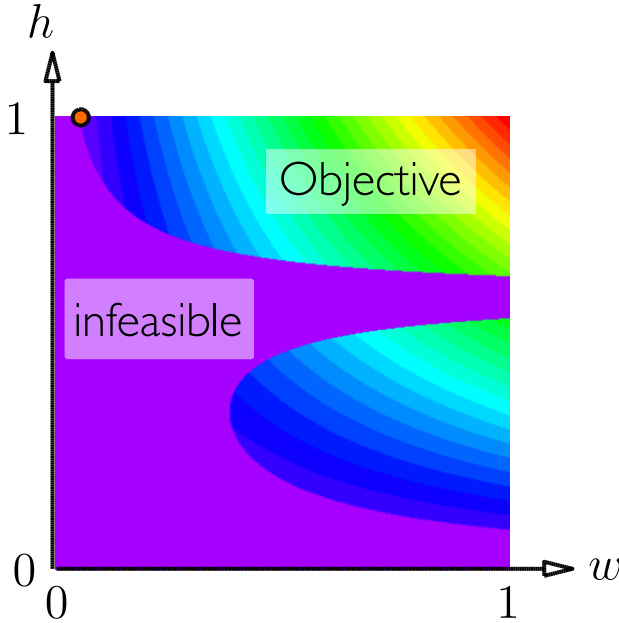
Figure 11: Beam with rectangular cross section

Figure 12: Deterministic objective and feasible design space. The deterministic optimal design is located in the red colored point ($d = 0.06, h = 1.00$).
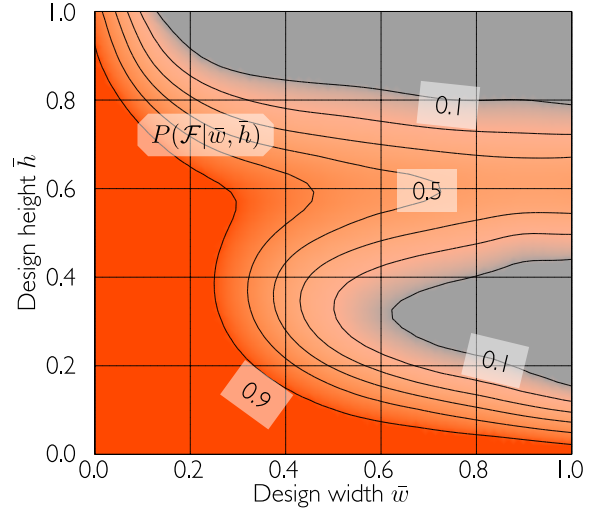
Figure 13: Compute conditional probability of violating constraint depending on h and d by using FORM.

is to be minimized subjected to deadload and a harmonic load. The central deflection w_d due to the dynamic load $F(t)$ has to be smaller than $5\,mm$. The computational probabilistic and multidisciplinary analysis tasks were done with the software package optiSLang (Bucher et al. (2001)). The objective function (i.e. the cross section area) and the admissible area are displayed in Figure 12 for assumed values of $F_0 = 20\,kN$, $\omega = 60\,rad/s$, $E = 3 \cdot 10^{10}\,N/m^2$, $\rho = 2500\,kg/m^3$, $L = 10\,m$ and $g = 9.81\,m/s^2$.

Furthermore, in many application cases – especially concerning structural dynamics – the characterizing parameters are afflicted with stochastic uncertainties. In the present example it is assumed that the dynamic load amplitude F_0 and the excitation angular frequency ω are random variables with Gaussian distribution. The mean values correspond to the aforementioned nominal values, and both variational coefficients have been assumed to be 10%. This yields that the restriction from the dynamic load can only be met with a certain probability < 1.
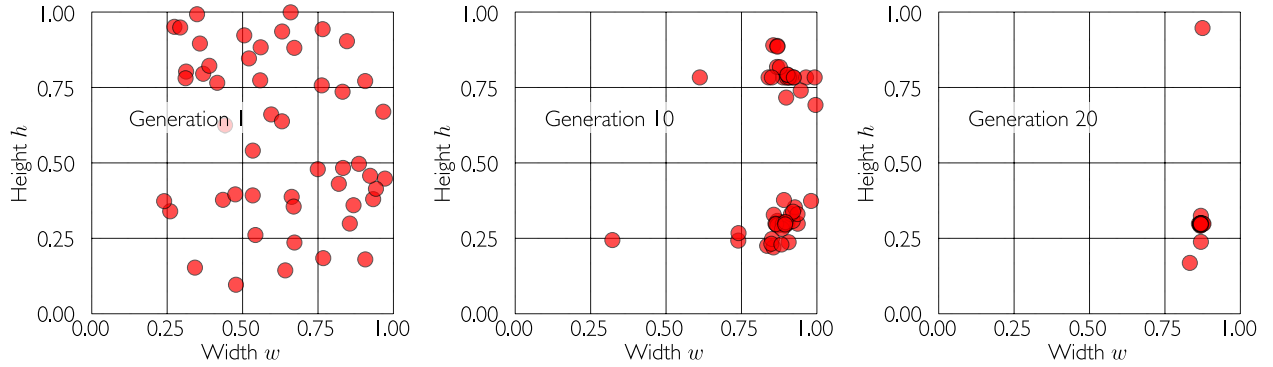
Figure 14: Evolutionary algorithm with 20 generations and 50 individual designs and FORM with 15 design evaluations ($N = 20 \cdot 50 \cdot 15 = 15.000$). Concentration of regions with acceptable failure probability. Best robust design: $d = 0.888$, $h = 0.289$ with a failure probability $0.98\% < 1\%$.

3.2 Evolutionary optimization with FORM

Fig. 12 shows that two separate admissible areas exist. A gradient based optimizer generally encounters difficulties to override the area boundaries in order to find the global optimum. Fig. 13 displays the probability of violation of the dynamic restriction (i.e. the conditional failure probability $P(\mathcal{F}|d,h)$) as a function of the design parameters d and h.

The subsequent optimization was started with the additional restriction that the conditional failure probability be $< 1\%$. In the framework of genetics, designs with a higher failure probability were punished by a penalty term S the value of which is independent from $P(\mathcal{F})$. Hence, the objective function writes

$$L = h \cdot d + SH[P(\mathcal{F} : w_d \leq 5\,mm) - 0.01] \tag{26}$$

In this equation $H[.]$ designates the Heavyside function. The penalty parameter S has been assumed as 100. A evolutionary optimization with 20 generation with 50 individuals each yielded the following best individual: The failure probability in this case was 0.98%, which is significantly below the threshold of 1%. Fig. 14 illustrates the progression of the genetic algorithm by displaying the populations of the first, the tenth, and the 20th generation. In addition, in order to calculate the failure probability using FORM for each individual design 15 design evaluations are necessary. So, in total $N = 20 \cdot 50 \cdot 15 = 15.000$ design evaluations to receive a concentration of designs on an area with acceptable failure probability. The cross sectional area was 0.26 which is considerably larger than in the value of 0.06 obtained in the deterministic case but the robust design optimization leads to more robust and safety design. So the user can adjust the design parameter to hit the target performance and to minimize the effect of variations and to minimize the failure probability.

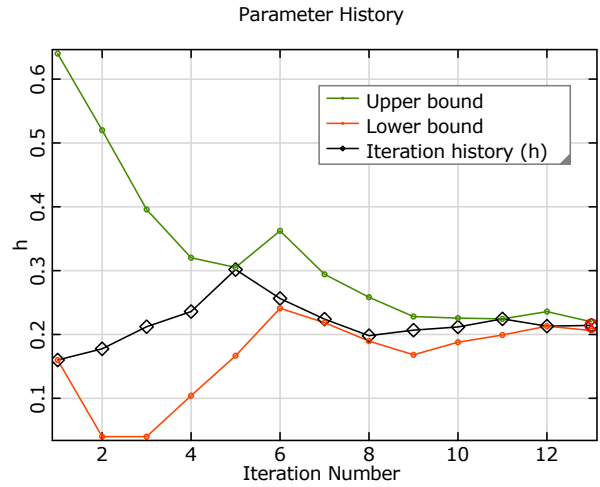## 3.3 Adaptive response surfaces

The approximation of the Heavyside function using MLS is possible in the abstract but the approximation accuracy can be improved using the objective function
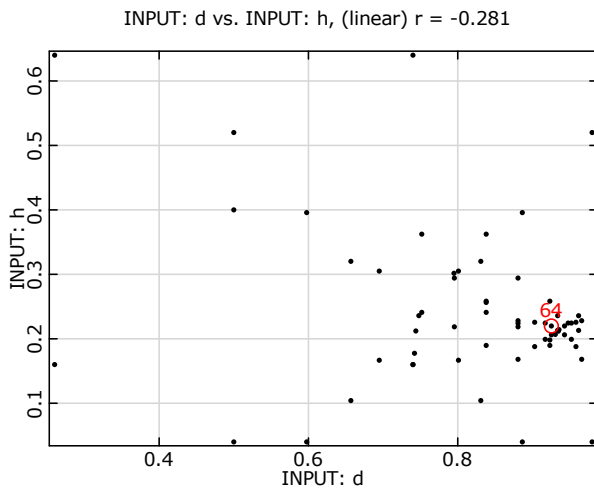
$$f(d, h) = h \cdot d$$
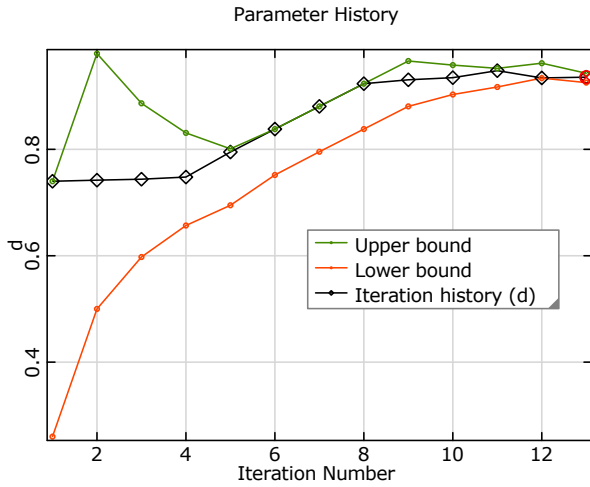
Figure 15: Adaptive modification of the design variable d and parameter bounds during the optimization.

Figure 16: Adaptive modification of the design variable h and parameter bounds during the optimization.

Figure 17: Adaptive design of experiment of design variable d and h during the optimization. The best robust design: $d = 0.925$, $h = 0.220$ with a failure probability $0.98\% < 1\%$.
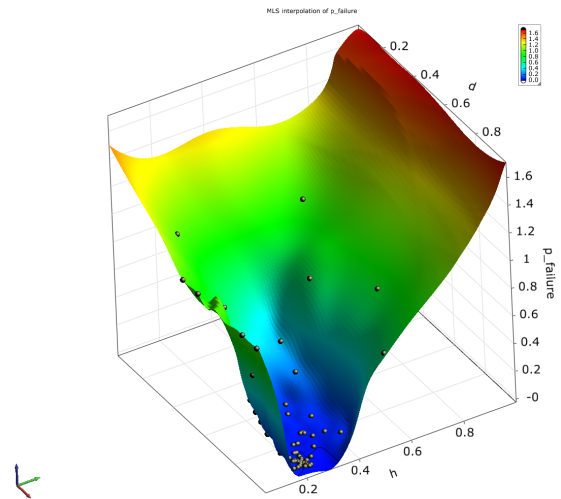
Figure 18: Approximation of the conditional probability of violating constraint depending on h and d by using ARSM.

and an additional stochastic constraint

$$P(\mathcal{F} : w_d \leq 5\,mm) - 0.01 \leq 0$$

instead equation 26. During the optimization procedure ARSM moves and shrinks the design parameter bounds, as shown in Fig. 15 and 16. Fig. 17 illustrates this adaptive design of experiments. In order to calculate the failure probability for each individual design using the ARSM proposed in Roos et al. (2006) additional 18 design evaluations on the space of the random variables F_0 and ω are necessary. So, in total $N = 64 \cdot 18 = 1152$ design evaluations to receive a design with minimal cross sectional area 0.20 and an acceptable failure probability of 0.0098% < 1%.

4   Concluding Remarks

Within the robust design optimization the design parameters can be random variables themselves and in addition the objective and the constraint functions can be random types. In case of the objective function includes the variances or other statistical data and the design variables are random one we obtain designs with minimal variance. Using the robust design optimization we obtain robust optimized designs such that they are insensitive to uncertainties within a safety level of two sigma. The reliability-based optimization includes the failure probability as constraint condition or as term of the objective function themselves. So we obtain design with minimal failure probability applicable for all safety levels up to 6 sigma.

Robust design optimization can provide multiple benefits. It permits the identification of those design parameters that are critical for the achievement of a certain performance characteristic. A proper adjustment of the thus identified parameters to hit the target performance is supported. This can significantly reduce product costs.

The effect of variations on the product behaviour and performance can be quantified. Moreover, robust design optimization can lead to a deeper understanding of the potential sources of variations. Hence, a minimization of the effect of variations (noise) is made possible, and appropriate steps to desensitize the design to these variations can be determined. Consequently, more robust and affordable product designs can be achieved.

The proposed evolutionary optimization combined with FORM is distinguished suitable for non-differentiable objectives and constraints and for searching for feasible design islands with a high number of design variables ($n_d > 100$). The point on the limit state function with the smallest distance to the origin is the most probable failure point or design point and can be found by gradient-based optimization. So the number of random variables should be lesser than $n_r < 50$. The gradient based optimization requires differentiable state functions and limit state functions but these cannot be presumed in general situations.

The introduced adaptive response surface method is suitable in cases of non-differentiable objectives and constraints and non-differentiable state functions and limit state functions. In contrast to a classical ARSM using ow level trial functions (e.g. linear and quadratic polynomial functions) the approximation is improved using additional support points within the MLS approximation. This approach is very efficient with D–optimal and linear DOE for a moderate number of design parameters $n_d \leq 20$ and random parameters $n_r \leq 20$.

References

S.J. Abspoel, L.F.P. Etman, J. Vervoort, R.A. van Rooij, A.J.G Schoofs, and J.E. Rooda. Simulation based optimization of stochastic systems with integer design variables by sequential multipoint linear approximation. *Structural and Multidisciplinary Optimization*, 22:125–138, 1996.

M. Allen and K. Maute. Reliability-based design optimization of aeroelastic structures. *Structural and Multidisciplinary Optimization*, 27:228 – 242, 2004.

Matthew Allen, Michael Raulli, Kurt Maute, and Dan M. Frangopol. Reliability-based analysis and design optimization of electrostaticallyactuated mems. *Computers and Structures*, 82(13–14):1007–1020, May 2004.

Aharon Ben-Tal and Arkadi Nemirovski. Robust optimization – methodology and applications. *Mathematical Programming (Series B)*, 92:453 – 480, 2002.

Juergen Branke. Creating roubust solutions by means of an evolutionary algorithm. *A.E. Eiben, T. Bäck, M. Schoenhauer, and H.-P. Schwefel, editors, ParallelProblem Solving from Nature*, 1498:119–128, 1998.

K. W. Breitung. Probability approximations by log likelihood maximization. *Journal of Engineering Mechanics, ASCE*, Vol. 117(No. 3):457 – 477, 1991.

C. Bucher, J. Will, and J. Riedel. Multidisciplinary non-linear optimization with optimizing structural language optislang. In *19th CAD-FEM USERS' Meeting 2001 International Congress on FEM Technology, October 17-19*, Postdam, Germany, 2001.

D.M. Byrne and S. Taguchi. The taguchi approach to parameter design. In *40th Annual Quality Congress Transactions*, pages 19 – 26. American Society for Quality Control, Milwaukee, Wisconsin, 1987.

Wei Chen, Jin Ruichen, and Agus Sudjianto. Analytical variance-based global sensitivity analysis in simulation-based design under uncertainty. In *Proceedings of DETC'04, ASME 2004 Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, Salt Lake City, Utah USA, September 28 – October 2 2004.

K. K. Choi, J. Tu, and Y. H. Park. Extensions of design potential concept for reliability-based design optimization to nonsmooth and extreme cases. *Structural and Multidisciplinary Optimization*, 22:335–350, 2001a.

K. K. Choi, Byeng D. Youn, and Ren-Jye Yang. Moving least square method for reliability-based design optimization. In *WCSMO-4*, Dalian, China, 2001b. Center for Computer-Aided Design and Department of Mechanical Engineering. URL http://www.icaen. uiowa.edu/~byoun/WCSMO-4.pdf.

Ioannis Doltsinis and Zhan Kang. Robust design of structures using optimization methods. *Comput. Methods Appl. Mech. Engrg.*, 193:2221 – 2237, 2004.

L.F.P. Etman, J.M.T.A. Adriaens, M.T.P. van Slagmaat, and A.J.G. Schoofs. Crashworthiness design optimization using multipoint sequential linear programming. *Structural Optimization*, 12:222–228, 1996.

L. J. Fogel, A. J. Owens, and M. J. Walsh. *Artificial intelligence through simulated evolution.* John Wiley & Sons, New York, 1966.

David E. Goldberg. *Genetic algorithms in search, optimization and machine learning.* Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1989. ISBN 0201157675.

A. M. Hasofer and N. C. Lind. Exact and invariant second-moment code format. *Journal of the Engineering Mechanics Division, ASCE*, 100(1):111–121, January/February 1974.

M. Hohenbichler and R. Rackwitz. Improvement of second-order reliability estimates by importance sampling. *Journal of Engineering Mechanics, ASCE*, Vol. 114, No. 12:2195 – 2198, 1988.

J. H. Holland. *Adaptation in natural and artificial systems.* University of Michigan Press, 1975.

K.-H. Hwang, K.-W. Lee, and G.-J. Park. Robust optimization of an automobile rearview mirror for vibration reduction. *Structural and Multidisciplinary Optimization*, 21:300– 308, 2001.

G. Kharmanda, A. Mohamed, and M. Lemaire. Efficient reliability-based design optimization using a hybrid space withapplication to finite element analysis. *Structural and Multidisciplinary Optimization*, 24:233 – 245, 2002.

P. N. Koch, R.-J. Yang, and L. Gu. Design for six sigma through robust optimization. *Structural and Multidisciplinary Optimization*, 26:235 – 248, 2004.

S. Kok and N. Stander. Optimization of a sheet metal forming process using successive multipoint approximations. *Structural Optimization*, 18(4):277–295, 1999.

H. Kurtaran, A. Eskandarian, D. Marzougui, and N.E. Bedewi. Crashworthiness design optimization using successive response surface approximations. *Computational Mechanics*, 29:409–421, 2002.

P. Lancaster and K. Salkauskas. *Curve and surface fitting; an introduction.* Academic Press, London, 1986.

R. E. Melchers. Optimality-criteria-based probabilistic structural design. *Structural and Multidisciplinary Optimization*, 23:34 – 39, 2001.

T. Most and C. G. Bucher. A moving least squares weighting function for the element-free galerkin method which almost fulfills essential boundary conditions. *Structural Engineering and Mechanics*, 21(3):315 – 332, 2005.

M.S. Phadke. *Quality Engineering using Robust Design.* Prentice Hall, Englewood Cliffs, New Jersey, 1989.

Olivier V. Pictet, Michel M. Dacorogna, Rakhal D. Davé, BatienChopard, Roberto Schirru, and Marco Tomassini. Genetic algorithms with collective sharing for robust optimization in financialapplications. OVP.1995-02-06, http://www.olsen.ch/research/307_ga_pase95.pdf, 1996.

R. Rackwitz and B. Fießler. Structural reliability under combined random load sequences. *Computers and Structures*, 9:489—494, 1978.

M. Rais-Rohani and M. N. Singh. Comparison of global and local response surface techniques in reliabilty-basedoptimization of composite structures. *Structural and Multidisciplinary Optimization*, 26:333 – 345, 2004.

I. Rechenberg. *Evolutionsstrategie: Optimierung technischer Systeme nach Prinzipien der biologischen Evolution.* Frommann-Holzboog, Stuttgart, 1973.

D. Roos, U. Adam, and V. Bayer. Desgin reliability analysis. In *24th CAD-FEM USERS' Meeting 2006 International Congress on FEM Technology, October 26-27*, Stuttgart, Schwabenlandhalle, Germany, 2006.

J. O. Royset and E. Polak. Reliability-based optimal design using sample average approximations. *Probabilistic Engineering Mechanics*, 19:331 – 343, 2004.

J. O. Royset, A. Der Kiureghian, and E. Polak. Successive approximations for the solution of optimal design problems withprobabilistic objective and constraints. In DerKiureghian, Madanat, and Pestana, editors, *Applications of Statistics and Probability in Civil Engineering*, pages 1049 – 1056. Millpress, Rotterdam, Ninth International Conference on Applications of Statistics and Probabilityin Civil Engineering (ICASP9), July 6-9 2003.

K. Schittkowski. Nlpql: A fortran subroutine for solving constrained nonlinear programming problems. *Ann Oper Res*, 5:485 – 500, 1985.

K. Schittkowski. Nlpqlp20: A fortran implementation of a sequential quadratic programming algorithm with distributed and non-monotone line search - user's guide. Report, Department of Computer Science, University of Bayreuth, Bayreuth, Germany, 2004.

H. P. Schwefel. *Numerical optimization of computer models.* John Wiley & Sons, New York, 1981.

M. Shinozuka. Basic analysis of structural safety. *Journal of Structural Engineering, ASCE*, Vol. 109(No. 3):721 – 740, 1983.

N. Stander and K.J. Craig. On the robustness of a simple domain reduction scheme for simulation-based optimization. *Eng. Comput.*, 19(4):431–50, 2002.

V. V. Toropov and L.F. Alvarez. Development of mars – multipoint approximation method based on the response surface fitting. Technical report, AIAA, 1998.

L. Tvedt. *Two second-order approximations to the failure probability.* Veritas Report R74-33, Det Norske Veritas, Oslo,Norway, 1983.

Benjamin Wilson, David Cappelleri, Timothy W. Simpson, and Mary Frecker. Efficient pareto frontier exploration using surrogate approximations. *Optimization and Engineering*, 2:31 – 50, 2001.

R. J. Yang and L. Gu. Experience with approximate reliabiltity-based optimization. *Structural and Multidisciplinary Optimization*, 26:152 – 159, 2004.

B. D. Youn, K. K. Choi, R. J. Yang, and L. Gu. Reliability-based design optimization for crashworthiness of vehicle sideimpacts. *Structural and Multidisciplinary Optimization*, 26:272 – 283, 2004.

Byeng D. Youn and Kyung K. Choi. A new response surface methodology for reliability-based design optimization. *Computers and Structures*, 82:241 – 256, 2004.