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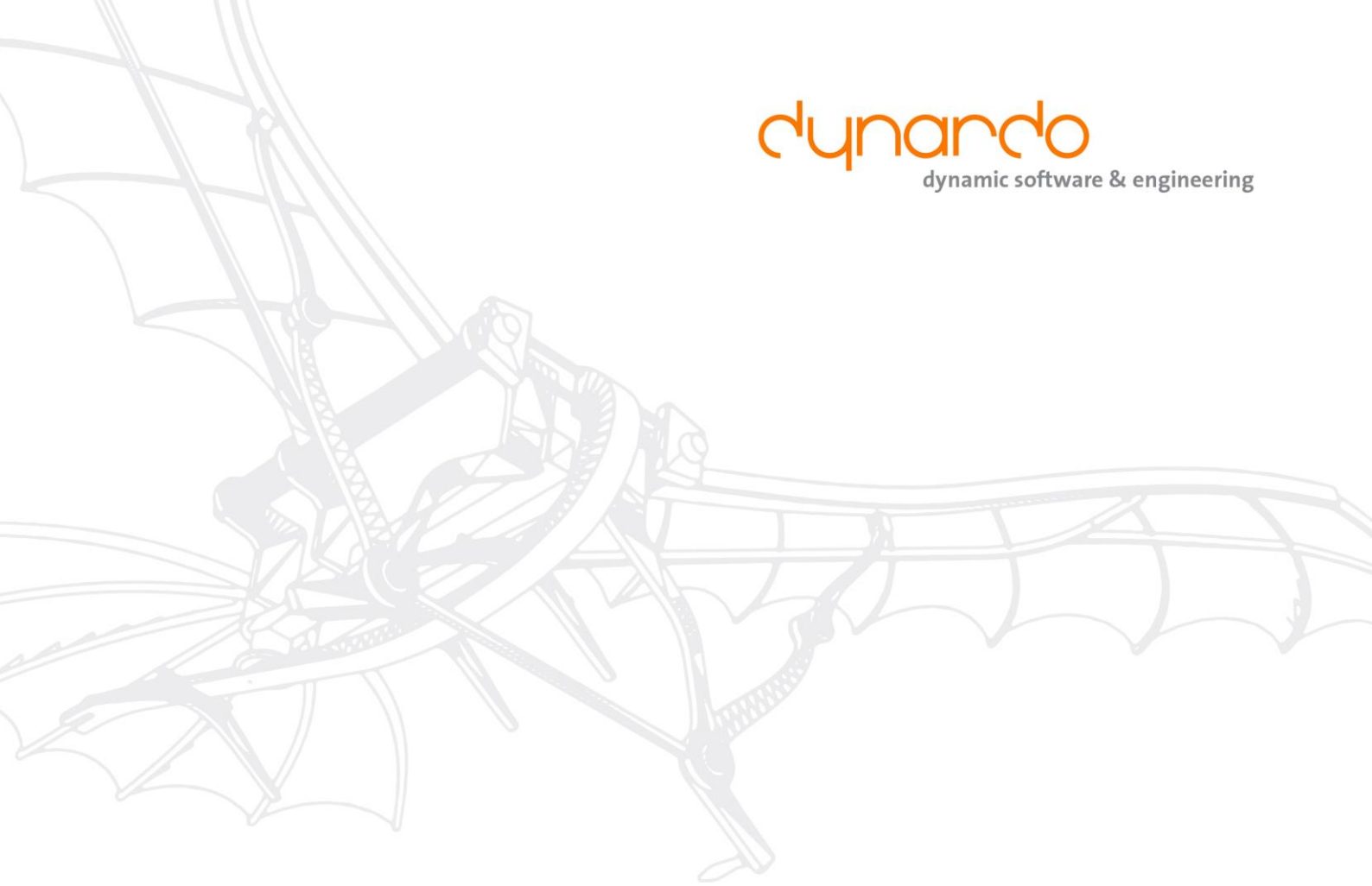

Lectures

Simulation of random fields in structural design

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Simulation of random fields in structural design

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Abstract: Many physical quantities in structural design are fields being distributed on structures and being random processes. The physical structures are often represented by finite element models, the fields are random variables defined at each mesh node. The analysis of the autocovariance function of such discrete random fields and their reduction to a limited number of mode shapes allow engineers to eliminate noise and, further, to identify and to quantify important parameters that influence the random field. This article deals with the opposite, i.e. the simulation of random fields by taking into account constraints (eg. lower and upper bounds of a yield stress), arbitrary random distribution functions and a spatial correlation structure. The presented algorithm is illustrated using the software package *Statistics on Structures* applied to an industrial example.

Keywords: random fields, simulation, geometric deviation, modal reduction, spatial correlation

1 Introduction

Current developments in CAE often employ investigations on robustness, sensitivity or statistics of random effects on finite element structures. For example, an optimization must be accompanied by a robustness and reliability analysis. For this purpose, random influences are measured from experiments or are generated by Monte Carlo methods in conjunction with FEM simulation [2]. The performance of a structure is then assessed by statistical means [8]. Many processes involve random quantities being spatially distributed on the examined structure.

When utilizing random fields one is able to assess random effects as well as their localization. They provide several levels of insight: First, the distribution of scatter on the structure is

observed and hot spots are located. Next, random field data can be decomposed into scatter shapes, which can be ranked by their contribution to the total scatter. These shapes can be used to reduce the number of random variables (which may be very large depending on the number of nodes and elements in the FEM mesh) and to reduce noise while only keeping the essential features of the random field.

DYNARDO [3] developed the software *Statistics on Structures* (SoS) which is capable of decomposing random fields into scatter shapes, analyzing random properties on FEM structures, locating "hot spots" of variation and investigating correlations. Aside the analysis of a given set of samples, engineers are also interested in generating random samples with a specified spatial correlation structure. For example, in sheet metal forming one often wants to generate random geometric imperfections (induced by the production process) and analyze the performance of the individual designs later on. This article deals with the simulation of pseudo-random fields focusing on very large structures. These simulations are usually restricted to small FEM meshes due to computer hardware limitations. Hence one needs to find suitable approximations which allow to generate random field samples with high numerical efficiency.

The article starts with an introduction to random fields and its Karhunen-Loeve expansion in section 2. Section 3 presents the brute force approach to simulation while section 4 enhances the procedure to very large meshes. Section 5 presents an industrial example illustrating the performance of the proposed methodology.

The author wants to point out that he originally used a different interpolation method for large scale FEM structures. During the review process, however, the reviewer proposed the usage of EOLE which turned out to outperform the original solution leading to major changes of the direction of this article, see section 4.1.

2 Random fields

Let us consider a scalar field which is defined by a real-valued function value H over an n-dimensional space, i.e.

$$H \,:\, \mathbb{R} \to \mathbb{R}^n \; \mathbf{x} \to z, \quad z \in \mathbb{R}, \, \mathbf{x} \in \mathbb{R}^n \tag{1}$$

In most structural applications this function is defined in three-dimensional space, i.e. $n = 3$. Let us assume that the field given through H usually varies smoothly in space whereby points being very close to each other have similar function values. If the field is influenced by random quantities, then one may measure different realizations of the same field, see figure 1. Therein, four random samples ω of the same scalar field H defined in one dimension x are illustrated. Two points in space are highlighted in the figure, for which one can compute statistical measures as if they are individual random variables. For example, one can compute the mean value function

$$\bar{H}(\mathbf{x}) = \mathbf{E}[H(\mathbf{x})] \tag{2}$$

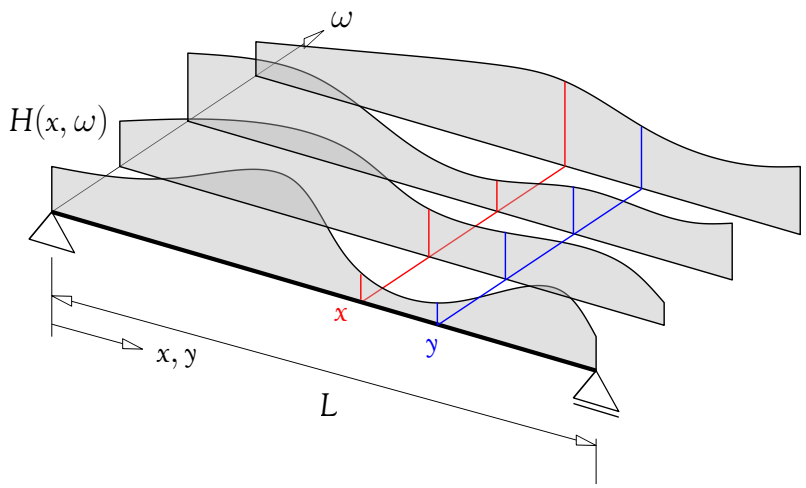


Fig. 1: Different realizations of a one-dimensional field. Source: [9]

One can further compute correlation measures for the field values at different coordinates, for example by the covariance between two points $\mathbf{x}$ and $\mathbf{y}$

$$C_{HH}(\mathbf{x},\mathbf{y}) = \mathbf{E}[\{H(\mathbf{x}) - \bar{H}(\mathbf{x})\}\{H(\mathbf{y}) - \bar{H}(\mathbf{y})\}] \tag{3}$$

with auto-covariance function C_{HH}.

The design space $\mathbf{x} \in \mathbb{R}^n$ is infinite large. In engineering applications one is interested in reducing the number of variables to a small finite number. A tool is to express the field H by a Fourier-type series expansion using deterministic basis functions ϕ_k and random coefficients c_k

$$H(\mathbf{x}) = \sum_{k=1}^{\infty} c_k \phi_k(\mathbf{x}), \quad c_k \in \mathbb{R},\ \phi_k \in \mathbb{R} \tag{4}$$

This transforms the field being originally expressed by the unknowns $\mathbf{x}$ to a space expressed by the unknowns $\mathbf{c}$. By truncating the series, a reduction of the number of variables can be achieved. The Karhunen-Loeve expansion states that an optimal choice of the basis functions is given by an eigenvalue ("spectral") decomposition of the auto-covariance function, i.e.

$$C_{HH} = \sum_{k=1}^{\infty} \lambda_k \phi_k(\mathbf{x})\phi_k(\mathbf{y}), \quad \int_{\mathbf{R}^n} C_{HH}(\mathbf{x},\mathbf{y})\phi_x(\mathbf{x})d\mathbf{x} = \lambda_k \phi_k(\mathbf{y}) \tag{5}$$

The obtained basis functions are orthogonal and the coefficients become uncorrelated.

When a scalar field is measured as a distribution on a FEM mesh (or on any other discrete space), the field H is represented by discrete values, i.e.

$$H_i = H(\mathbf{x}_i), \quad i = 1 \ldots N \tag{6}$$

In this case the spectral decomposition is given through

$$H_i - E(H_i) = \sum_{k=1}^{N} \phi_k(\mathbf{x}_i) c_k = \sum_{k=1}^{N} \phi_{ik} c_k \tag{7}$$

or in matrix-vector notation

$$\mathbf{H} = \Phi\mathbf{c} + \bar{\mathbf{H}} \tag{8}$$

Again, a significant reduction in the number of variables can be achieved when truncating the series after a few items. The field H being measured in terms of a large number of values H_i (usually in terms of single values per node or finite element) is expressed through a small number of coefficients c_k. The "scatter shapes" ϕ_{ik} define the transformation basis.

By reducing the number of random variables, one improves the statistical significance for a small sample size (eliminates noise), reduces the numerical effort in statistical analysis and may simplify the representation of input/output relations based on meta models. The basis functions should be orthogonal reducing the computational effort for the projection (reduction) and its inverse transformation. As a side effect the random coefficients are uncorrelated simplifying the digital simulation of random fields.

3 Simulation of random fields

For simplicity, the consideration is restricted to the following assumptions:

- The distribution type (with CDF $F(X_j)$) of individual points in the field is the same for all points j.
- The realization of a random variable may be bounded (due to restrictions in the production process).
- Inhomogenity is limited to different mean and standard deviation at individual field points.
- The spatial correlation is defined by autocorrelation functions, i.e. $C(d) = C_{HH}(\mathbf{x}, \mathbf{y})$ wherein $d = d(\mathbf{x}, \mathbf{y})$ denotes the distance between two points. (eg. through correlation length parameters).
- The spatial correlation is defined in standard-normal space.

3.1 Very small correlation length

If a very small correlation length is chosen, one can assemble a sparse correlation matrix using piecewise polynomial covariance functions [7], i.e.

$$C_{l,p}(d) = (1 - d/l)_+^p\,, \quad p > 2 \tag{9}$$

with correlation length l and polynomial order p. The covariance matrix Cov_{ij} defining the covariance between two mesh points $\mathbf{x}_i$ and $\mathbf{x}_j$ has compact support. The simulation then follows the steps:

1. Perform a decomposition of the covariance matrix $\mathbf{Cov} = \mathbf{L}\mathbf{L}^T$, eg. by a sparse Cholesky factorization.

2. Simulate N field vectors $\mathbf{u}_k$ of statistically independent standard-normal random variables, one number for each node.

3. Apply the correlation in standard normal space for each sample k: $\mathbf{z}_k = \mathbf{L}\mathbf{u}_k$.

4. Transform the correlated field samples into the space of the desired random field: $x_{k,i} = F^{(-1)}\left(N(z_{k,i})\right)$.

This procedure allows the generation of random field samples without approximation or loss of model information.

The major challenge is the numerical efficiency of the Cholesky factorization. In numerical tests using a mesh with only 33.000 nodes, a correlation length of 5% of the largest model dimension already leads to more than 1% non-zero elements in the covariance matrix. In turn the simulation took more than half an hour on a recent Intel i7 processor (using the Eigen sparse solver http://eigen.tuxfamily.org). In practice, in particular when simulating geometric imperfections, a correlation length of 20% - 50% is often used making this approach unattractive for moderately large meshes and impossible for very large meshes.

3.2 Large correlation length

For a large characteristic length-scale one usually obtains a dense covariance matrix. A widely used covariance function is the squared exponential function, i.e.

$$C_l(d) = \exp\left(-\frac{d^2}{2l^2}\right) \tag{10}$$

with correlation length l. A spectral decomposition is used to factorize the covariance matrix by $\mathbf{Cov} = \Phi\text{diag}(\lambda_{ii})\Phi^T$ with eigenvalues λ_i and orthogonal eigenvectors $\Phi = [\phi_i]$. This decomposition is used to reduce the number of random variables. Given a moderately large correlation length, only a few (eg. 3-5) eigenvectors are required to represent more than 90% of the total variability.

The simulation then follows the steps:

1. Perform a decomposition of the covariance matrix $\mathbf{Cov} = \Phi\text{diag}(\lambda_{ii})\Phi^T$ and chose c basis vectors ϕ_i being associated with the largest eigenvalues.

2. Simulate N vectors $\mathbf{u}_k$ of statistically independent standard-normal random variables, each vector is of dimension c.

3. Apply the correlation in standard normal space for each sample k: $\mathbf{z}_k = \sum_i^c \sqrt{\lambda_i}\phi_i u_{k,i}$.

4. Transform the correlated field samples into the space of the desired random field: $x_{k,j} = F^{(-1)}\left(N(z_{k,j})\right)$.

A global error measure ϵ may be based on the total variability being explained by the selected eigenvalues, i.e.

$$\epsilon = 1 - \frac{\sum_{i=1} c\lambda_i}{\sum_{i=1} n\lambda_i} = 1 - \frac{1}{n}\sum_{i=1} c\lambda_i \tag{11}$$

wherein n is the number of discrete points.

This procedure allows the generation of random field samples with relatively large correlation length parameters. It is based on a model order reduction, i.e. only a portion of the desired variability can be retained. Furthermore, the covariance matrix is stored as a dense matrix. Hence, the size of the FEM mesh is effectively limited to ≈ 10.000 nodes.

4 Simulation of random fields on large scale structures

An enhancement of the strategy presented in 3.2 can be used to generate random field samples on very large FEM meshes. The idea is to select an appropriate subspace. The covariance matrix is created in this sub space, the same is true for the generation of spatially correlated standard normal variables. Then one must interpolate the missing field points in the original space based on the sub-space samples. This interpolation must be chosen in such a way that it optimally represents the statistical properties of the random distribution at the respective field points.

For the choice of the sub-space one may use a second mesh with small number of nodes approximating the fine original FEM mesh. Such a strategy requires the implementation of a mesh coarsening algorithm. Such an algorithm may be difficult to implement in practice, if sufficiently accurate coarsening is aimed at for a great variety of mesh topologies (bold vs. thin-walled structures, multiple finite element types - 1d, 2d, 3d, 2.5d, etc.).

In this article, a very simple and efficient strategy is used to select the sub-space: The node indices are randomly chosen from the original FEM mesh.

The simulation then follows the steps:

1. Randomly select M support points from the finite element mesh.

2. Assemble the covariance matrix for the selected sub-space.

3. Perform a decomposition of the covariance matrix $\mathbf{Cov} = \Phi \text{diag}(\lambda_{ii}) \Phi^T$ and chose c basis vectors ϕ_i.

4. Create basis vectors ψ_i by interpolating the values of $\sqrt{\lambda_i}\phi_i$ on the FEM mesh.

5. Simulate N vectors $\mathbf{u}_k$ of statistically independent standard-normal random variables, each vector is of dimension c.

6. Apply the correlation in standard normal space for each sample k: $\mathbf{z}_k = \sum_i^c \psi_i u_{k,i}$.

7. Transform the correlated field samples into the space of the desired random field: $x_{k,j} = F^{(-1)}\left(N(z_{k,j})\right)$.

4.1 Expansion Optimal Linear Estimation (EOLE)

An optimal representation of a stochastic field as a linear combination of a subset of its values was proposed by [4]. It is widely used when simulating random fields, see for example [6, 1] and the references therein. The method is called Expansion Optimal Linear Estimation (EOLE) and is an extension of Kriging. Kriging interpolates a random field based on samples being measured at a sub-set of mesh points. It then minimizes the variance of the interpolation error. When simulating random fields, one is interested in minimizing the variance of the error when representing the stochastic field by the Karhunen-Loeve expansion while only the base vectors are known at a sub-set of field nodes.

Assume that the sub-space is described by the field values

$$\mathbf{y}_k = \{z_{k,1}, \ldots, z_{k,M}\} = \{\sum_{i=1}^{c} \sqrt{\lambda_i}\phi_{i,1}u_{k,i}, \ldots, \sum_{i=1}^{c} \sqrt{\lambda_i}\phi_{i,M}u_{k,i}\} \tag{12}$$

Minimization of the variance between the target random field and its approximation under the constraint of equal mean values of both results in:

$$\psi_i = \mathbf{Cov}_{zy}^T \mathbf{Cov}_{yy}^{-1} \frac{\phi_i}{\sqrt{\lambda_i}} \tag{13}$$

with $\mathbf{Cov}_{yy}$ denoting the correlation matrix between the sub-space points and $\mathbf{Cov}_{zy}$ denoting the (rectangular) covariance matrix between the sub-space points and the nodes in full space. For the choice of an optimal number of required sub-space points per correlation length, one could further read [5].

4.2 Restoring statistical moments

The mean estimate $E(\hat{\mathbf{X}})$ of the generated random field samples $\mathbf{x}_k$ is unbiased, i.e. $E(\mathbf{X}) = E(\hat{\mathbf{X}})$. The variance, however, is bounded, i.e. $Var(\mathbf{X}) \geq Var(\hat{\mathbf{X}})$. This results directly by application of two smoothening filters, the modal reduction of the covariance matrix and the interpolation of the sub-space field samples. Both eliminate parts of the original variability which will not be present in the final samples.

In order to restore the desired variance, one may reintroduce the eliminated random parts of the solution in terms of white noise. To achieve this, statistically independent random numbers $e_{k,j}$ are generated for each sample and for each field point and added to the field sample vectors before the transformation into the target space takes place. The mean of these random numbers is zero while the standard deviation can be computed from

$$Z_j = Z_{j,approx} + Z_{j,error} \tag{14}$$

$$Var(Z_j) = E\left[\left(\sum_{j}^{c} \psi_j U_j + Z_{j,error}\right)^2\right] = 1 \tag{15}$$

$$\sigma_j^2 = 1 - \sum_{j}^{c} \psi_j^2 \tag{16}$$

Additionally to the steps in the previous section, one generates statistically independent standard-normal random numbers $e_{j,k}$ for each sample k and for each field point j. The field samples are then computed from $x_{k,j}^{corr} = F^{(-1)}\left(N\left(z_{k,j} + \sigma_j e_{j,k}\right)\right)$.

Furthermore, the vector σ provides a measure for the actual local error arising from the two steps of model order reduction. A global error can be obtained by relating the sum of the eigenvalues of the actual correlation matrix to the sum of the eigenvalues of the true correlation matrix. One first computes the eigenvalues κ_i of $\Psi\Psi^T$ (the nonzero eigenvalues of the matrix $\Psi^T\Psi$ are the same, but more efficient to compute). Then the global error is

$$\epsilon = 1 - \frac{1}{n}\sum_i \kappa_i \tag{17}$$

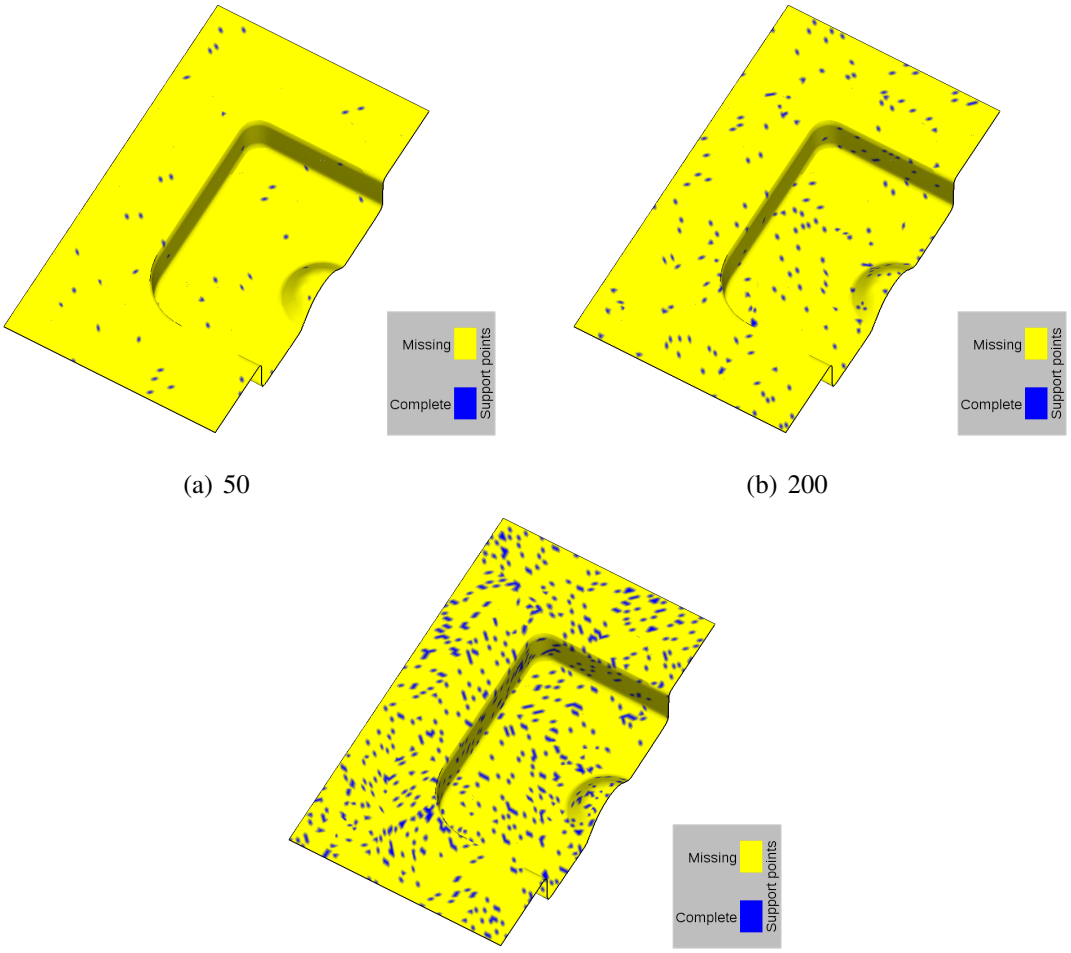
wherein n is the number of all discrete points.

5 Example

A simple structure from a sheet metal forming application serves as a numerical example. It is modelled by 4-node shell elements using 8786 finite element nodes. The maximum dimension is 540. The correlation length parameter is chosen to be 100 in standard normal space. The distribution of the considered field quantity (eg. a geometric deviation due to random process parameters) is a truncated Gaussian distribution with mean value 5, standard deviation 15, lower bound -20 and upper bound 30 - being equal at all nodes. The number of nodes is small enough to compute a reference solution based on the full model. The sub-space dimension is chosen to be small (between 50 and 1000 points). Then one may extrapolate the results to the large-scale case of > 50.000 nodes.

The correlation matrix is expanded using 20 eigenvectors representing $> 99.9\%$ variability in sub-space. Figure 2 shows the support point coordinates using different numbers of support points in sub-space. Figure 3 shows the first shapes ψ_i obtained from 50 support points. For comparison, figure 4 presents the same shapes if the correlation matrix is computed in full space. The MAC values comparing the full model with individual sub-space shapes are listed in table 1. The table also lists the global error and the average local error. The average local error is approximately the the square root of the global error norm obtained by the eigenvalues κ_i. The MAC values agree well even for very small sub-space dimensions. Figure 5 shows the corresponding local approximation errors of the spatial correlation. Surprisingly, the EOLE method approximates the problem very well even for a very small number of support points.

The local statistics is evaluated at all nodes, i.e. the mean value $\bar{X}$, the standard deviation σ_X and the 99% quantile value $F^{-1}(0.99) = 39.8952$ agree with the respective estimators of the simulated random numbers.



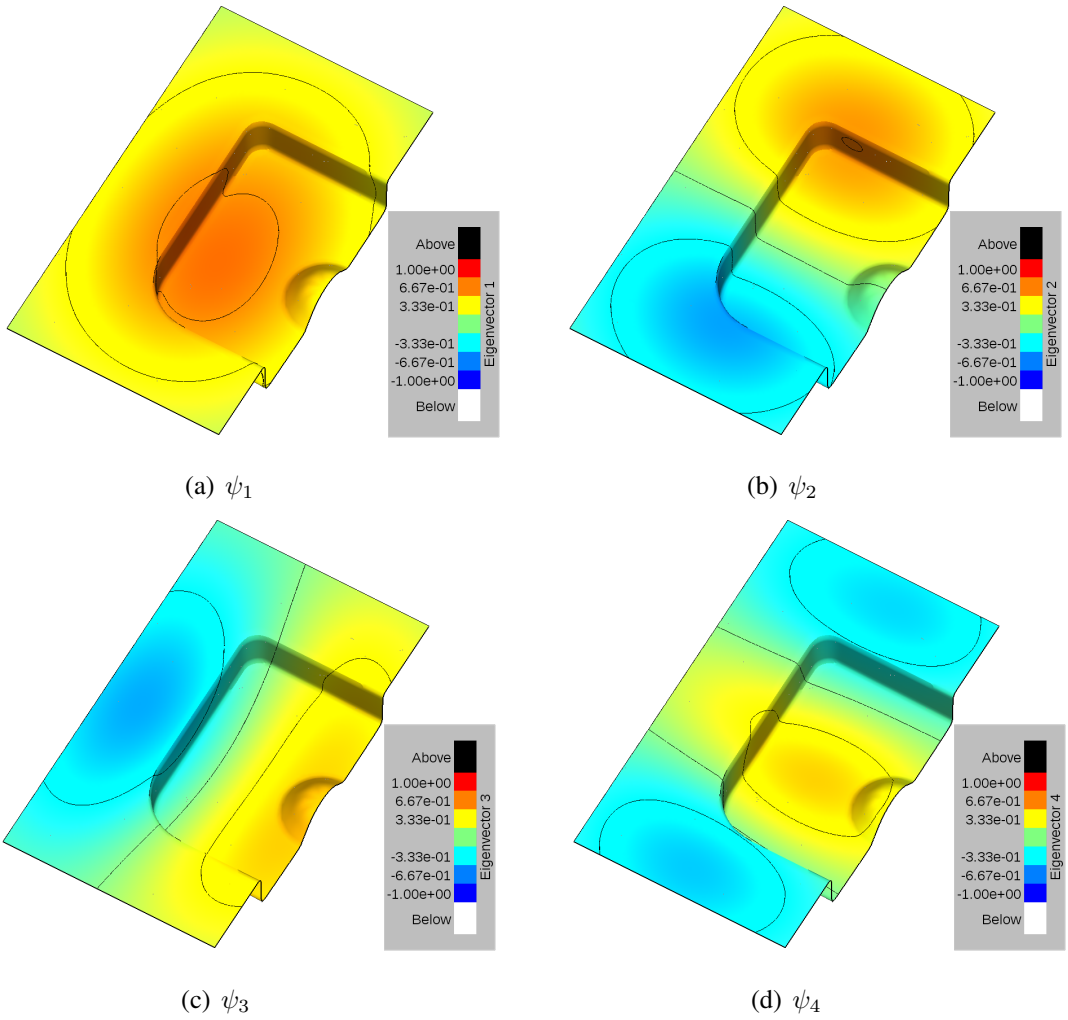
(a) 50 (b) 200

(c) 800

Fig. 2: Selected support point positions for different numbers of support points

Tab. 1: Errors and MAC values of various shapes (reference of comparison: full model) for different numbers of support points n.

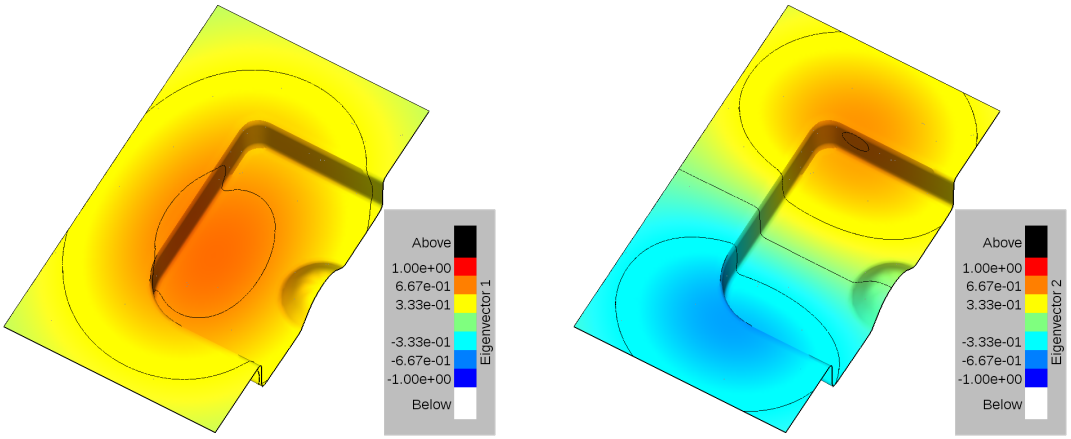
n	global error	mean local error	MAC ψ_1	MAC ψ_2	MAC ψ_5	MAC ψ_{10}
50	0.055	0.185	0.999	0.999	0.949	0.393
100	0.027	0.151	0.999	0.999	0.999	0.986
200	0.024	0.145	0.999	0.999	0.999	0.999
400	0.023	0.141	0.999	0.999	0.999	0.999
800	0.022	0.140	1	1	0.999	0.999
8786	0.021	0.140	1	1	1	1



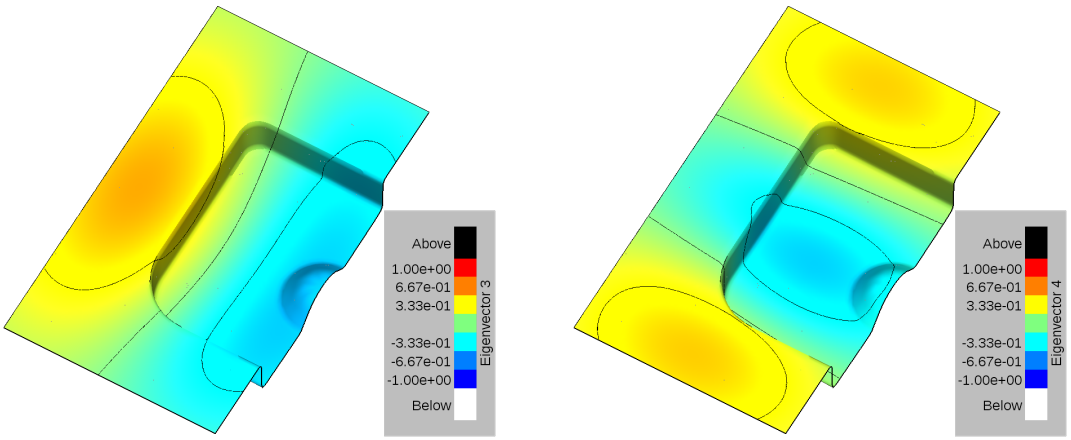
(a) ψ_1 (b) ψ_2

(c) ψ_3 (d) ψ_4

Fig. 3: First shapes ψ_k for 50 support points in sub-space

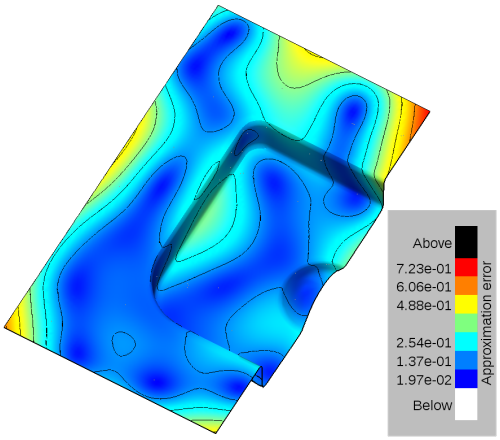


(a) ψ_1 (b) ψ_2

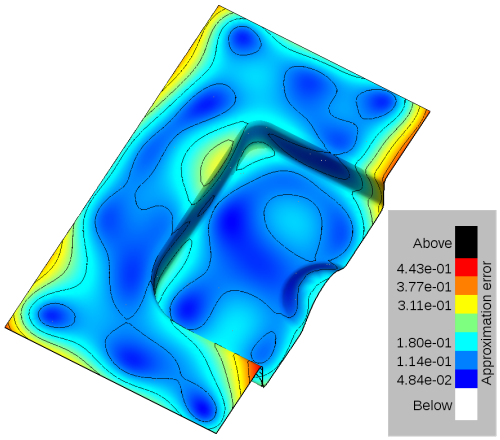


(c) ψ_3 (d) ψ_4

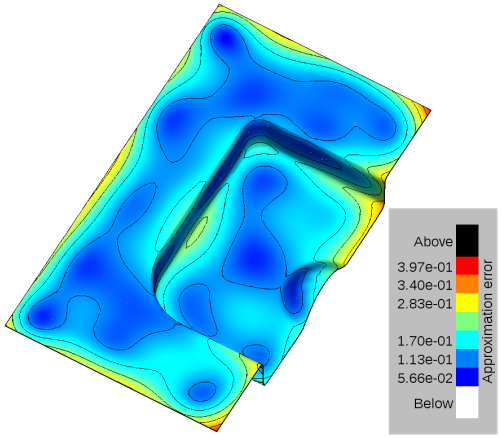
Fig. 4: First shapes ψ_k using the full model



(a) 50



(b) 200



(c) 800

Fig. 5: Local approximation error for different numbers of support points

6 Conclusion

This article presented two strategies to simulate random fields on large finite element structures. The strategy for very small correlation length-scale parameters based on a sparse correlation matrix is not yet applicable in practice due to the missing availability of fast numerical linear algebra packages that provide access to the Cholesky factors of a sparse matrix. It is, however, possible to efficiently generate random fields with relatively large correlation length-scale parameters based on approximation methods. Herein, a spectral decomposition of the covariance matrix in a randomly selected sub-space and interpolation in full space using the EOLE method was studied. Errors due to the approximation can be easily quantified. The spatial correlation structure may be well represented even if only a very small number of support points is used. The presented algorithms will be soon available in the software *Statistics on Structures* by DYNARDO Austria GmbH.

Acknowledgement

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