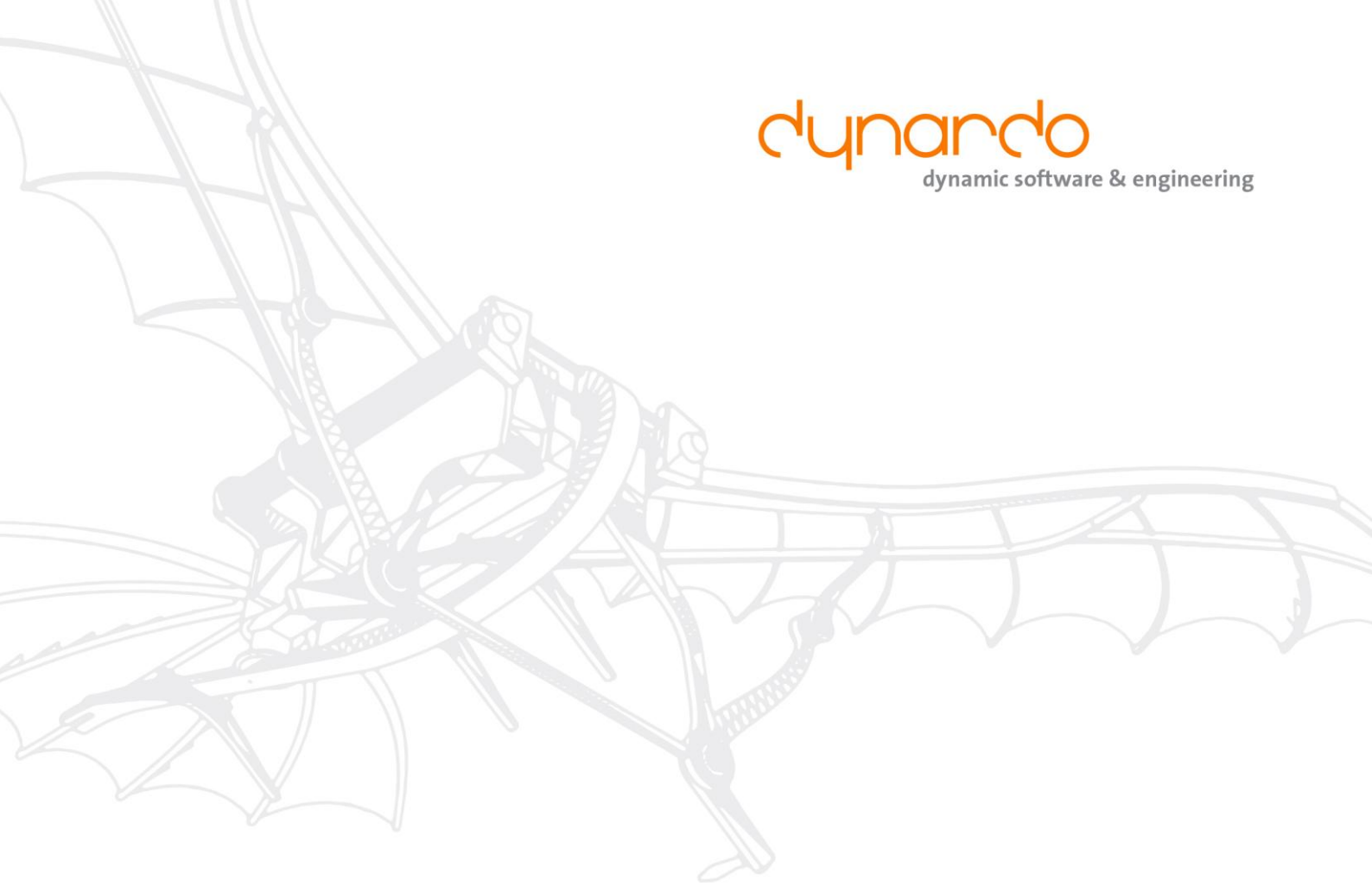
dynardo
dynamic software & engineering



Lectures

Structural Reliability Analysis by Random Field Modeling with Robustness Methods and Adaptive Response Surfaces

V.Bayer, D.Roos & U.Adam

presented at the Int. Conf. on Civil, Structural and Environmental Engineering Computing 2007
Source: www.dynardo.de/en/library

to appear: 11th Int. Conf. on Civil, Structural and Environmental Engineering Computing
St. Julians, Malta, 18–21 September 2007

Structural Reliability Analysis by Random Field Modeling with Robustness Methods and Adaptive Response Surfaces

V. Bayer, D. Roos & U. Adam
Dynardo - dynamic software & engineering GmbH, Weimar, Germany

Abstract

Any mechanical or civil engineering structure possesses some natural randomness in its properties which fluctuates over space. These can be modelled as random fields for the purpose of robust optimization or reliability analysis. The high number of random variables required to model a random field often inhibits accurate probabilistic analyses based on Monte Carlo methods.

The present paper proposes a method to reduce the random space based on both stochastic criteria and structural performance. This way the most relevant variables are identified in order to be used in a subsequent reliability analysis.

For computation of reliablity, an adaptive response surface method is introduced, which utilizes improved Moving Least Squares approximations. It is able to model highly nonlinear limit state functions and can be locally refined.

The stability of a cylindrical shell with random geometry is studied. A conventional and the proposed new way to reduce the random space are compared. The structure is then analysed by the adaptive response surface method, results are compared to direct directional simualtion.

The probabilistic and structural analyses are performed with the optiSLang, SoS and SLang software packages.

Keywords: reliability analysis, random fields, stochastic finite elements, robustness analysis, response surfaces, moving least squares.

1 Introduction

Any mechanical or civil engineering structure possesses some natural randomness in its properties which fluctuates over space: deviations from the design geometry, surface roughness, scatter of material properties and distributed loads are few examples. There exist attempts to model uncertainties by few random variables which are generated by CAD programs. This approach is meaningful for few special problems only. For a realistic interpretation of such random fluctuations within an optimization, robustness or reliability analysis, they have to be modelled as random fields. To distinguish the random field approach from the previous one, it is called it non–parametric here.

The stochastic analysis software SLang (which stands for: Structural Language) includes several methods to solve all of the above stochastic models. Currently, optiSLang (the optimizing Structural Language) supports methods to analyse random variables only. It offers additional functions for robustness evaluation and structural optimization. In addition, the SoS (Statistics on Structures) add-on tool to optiSLang provides methods to solve random fields.

In a computer implementation, random processes or fields are discretized into a finite set of random variables. The number of variables, however, can be considerably high. Most of the accurate methods for computing structural reliability (such as advanced Monte Carlo Methods) as well as dependable approximate methods (e.g., the Response Surface Method) are sensitive towards the number of variables involved. The present paper proposes a method to model a random field with a reduced set of variables. Robustness evaluation (section 3) is employed for this purpose, which relates the stochastic input and output quantities and thus helps to identify the most relevant variables. Unlike previous approaches (sketched in sec. 2.2), it does not rely on purely stochastic considerations, but takes into account the structural behaviour as well.

This reduced set is then utilized for a reliability analysis. Although Monte Carlo methods are most versatile, intuitively clear, and well understood ([18], [34], see [1] for an overview), the computational cost is in many cases prohibitive, when the limit state is formulated implicitely, i.e. evaluated by a Finite Element analysis. Hence approximations become important which can be based e.g. on the response surface method (RSM) [3, 15, 28]. It aims at a global approximation of the limit state, which is formally simple and fast to evaluate, such that Monte Carlo simulation can be applied successfully.

The RSM procedure presented here uses an improved Moving Least Squares approximating functional. It meets the support points exactly. Moreover, additional supports can be added within an automatic adaptation in order to provide a good local approximation in important regions.

As an example, the reliability of a cylindrical shell structure with random imperfections is studied. Within this example, the imperfections are discretized by Stochastic Finite Element methods [5, 11, 17]. It is demonstrated, how Robustness Analysis

is employed in order to identify the most relevant random variables. The probability of failure is computed by Directional Sampling [2, 12, 26] as a reference, and by the Adaptive RSM in connection with Adaptive Monte Carlo [7].

2 Random Fields

2.1 Properties

A random field is, in brief, a random function $H(\mathbf{r})$ defined on a spatial structure. The vector $\mathbf{r} \in \mathbb{R}_{Str.}$ points to a location on the structure. Random fields are used, e.g., to study random fluctuations in geometrical or material properties of a mechanical or structural system. In other words, the considered property is a random variable at each point on the structure. Moreover, the random properties at two different locations can be mutually correlated among each others.

Any random variable is characterized by a probability distribution function, which can be parameterized by distribution type and stochastic moments. For random fields, the moments become functions over space as well. *From now on, a Gaussian (or Normal) distribution type is assumed.* In this case, the characterization by first and second moments provides the full information. In particular

$$\mu_H(\mathbf{r}) = \mathrm{E}[H(\mathbf{r})] = \int_{-\infty}^{+\infty} h f_H(\mathbf{r}, h)\, dh \tag{1}$$

denotes the *mean function*, and

$$R_{HH}(\mathbf{r}_1, \mathbf{r}_2) = \mathrm{E}[H(\mathbf{r}_1) \cdot H(\mathbf{r}_2)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_1 h_2 f_H(\mathbf{r}_1, \mathbf{r}_2, h_1, h_2)\, dh_1\, dh_2 \tag{2}$$

the *correlation function*, with $\mathrm{E}[.]$ being the expected value operation. R_{HH} is a function of the distance between two points and indicates the amount of linear dependency between the random properties at these locations.

The so-called *correlation length* L_{HH}, which is actually the centre of gravity of the correlation function, is a typical characteristic of R_{HH}. It has to be estimated from manufacturing processes, natural scatter of material properties, etc. An infinite correlation length yields a structure with random properties, yet without fluctuations within the structure. A zero correlation length yields uncorrelated (in case of the Gaussian distribution independent) variables.

Three special cases are important for the further studies. *Homogeneity:* A random field is said to be homogeneous in the wide sense, if the first and second moments are the same at any possible location, i.e.

$$\mu_H(\mathbf{r}) = \mu_H(\mathbf{r} + \boldsymbol{\xi}) \quad \forall\, \boldsymbol{\xi} \tag{3}$$

$$R_{HH}(\mathbf{r}_1, \mathbf{r}_2) = R_{HH}(\mathbf{r}_1 + \boldsymbol{\xi}, \mathbf{r}_2 + \boldsymbol{\xi}) \quad \forall\, \boldsymbol{\xi} \tag{4}$$

Isotropy (in the wide sense) claims that the correlation function depends on the distance between the two observed locations $\mathbf{r}_1,\ \mathbf{r}_2$ only, not on the direction:

$$R_{HH}(\mathbf{r}, \mathbf{r} + \boldsymbol{\xi}) = R_{HH}(\|\boldsymbol{\xi}\|) \tag{5}$$

In case of *orthotropy* the correlation function is a product of two or more independent functions defined on orthogonal axes:

$$R_{HH}(\mathbf{x}, \mathbf{y}) = R(\mathbf{x}) \cdot R(\mathbf{y}) \tag{6}$$

2.2 Modelling

For computational analyses, a random field has to be discretized in order to yield a finite set of random variables $\mathbf{X}$, which are assigned to discrete locations on the observed structure. Since the Finite Element Method is the standard for structural analyses, it is convenient to discretize the random field in the same way as the finite element model. One speaks of Stochastic Finite Elements in this case. The discretization can be oriented at the element mid points, integration points, or nodes. The properties of the random variables are derived from the random field properties explained previously. The spatial discretization should be able to model the variability of the random field. For this purpose, it has been recommended by Der Kiureghian and Ke [11], Hisada and Nakagiri [19] that the distance between two discretization points should be not more than $1/4$ of L_{HH}.

The set of random variables is then characterized by a mean vector and a correlation matrix. It is convenient for the developments that follow to use the covariance matrix instead, which is defined as

$$\mathbf{C}_{XX}: \quad c_{ij} = R_{HH}(\mathbf{r}_i, \mathbf{r}_j) - \mu_H(\mathbf{r}_i) \cdot \mu_H(\mathbf{r}_j) \tag{7}$$

The joint density of all random variables can be modelled with help of the Nataf model [24, 30], given the type and properties of the marginal distributions for each variable.

From now on, *random fields with zero mean vector are considered.* Then the covariance matrix suffices for the characterization of the random variables set. Random number generators can produce independent random variables only. For the assumed case of Gaussian distributed variables, independence is equivalent to zero correlation. It can be shown that the random variables will be uncorrelated after the following transformation. The covariance matrix is decomposed with help of an eigenvalue analysis:

$$\boldsymbol{\Psi}^T \mathbf{C}_{XX} \boldsymbol{\Psi} = \mathrm{diag}\{\lambda_i\} \tag{8}$$

Therein, $\boldsymbol{\Psi}$ is the matrix of eigenvectors of $\mathbf{C}_{XX}$ stored columnwise, and the eigenvalues are identical to the variances of the uncorrelated random variables Y_i: $\lambda_i = \sigma^2_{Y_i}$. The transformation rule reads

$$\mathbf{Y} = \boldsymbol{\Psi}^T \mathbf{X} \tag{9}$$

and the backward transformation

$$\mathbf{X} = \mathbf{\Psi} \ \mathbf{Y} \tag{10}$$

because the eigenvectors $\mathbf{\Psi}$ form an orthonormal basis. Hence it is possible to model the random field by a Karhunen-Loève expansion of the random field [17] which consists of a sum of deterministic shape funtions $\mathbf{\Psi}$ multiplied by the respective uncorrelated random amplitudes $\mathbf{Y}$.

The eigenvalues are usually stored sorted by magnitude in descending order, which is a measure for their contribution to representing $\mathbf{C}_{XX}$. This opens a way of reducing the usually huge number of variables. Only the random variables with the highest variances are employed for a later Monte Carlo simulation. The quality of approximation of the random field is expressed by the variability fraction [5]

$$Q = \frac{\sum_{i=1}^{n} \sigma_{Y_i}^2}{\mathrm{trace}\big(\mathbf{C}_{XX}\big)} \ ; \quad 0 \leq Q \leq 1 \tag{11}$$

The number of the random variables considered has to be adjusted before the simulation in order to reach a sufficient quality, e.g. $Q > 0.9$. However, although the random field approximation seems good, it may be not suitable for a reliability analysis. Schorling [35] reported this experience for the application to stability analysis of a randomly imperfect shell structure. The truncated series explained above is not suitable, because the criterion is purely stochastic and does not account for the structural behaviour. On the other hand, purely mechanical considerations may not work as well, if they fail to represent the random field properly.

This is the motivation for the development of a procedure which selects those mode shapes and random variables, which contribute "most" – by means of stochastic influence – to a certain structural performance, here the buckling load. The Robustness Analysis, as described briefly in the following, offers tools for this purpose.

3 Robustness Analysis

The program optiSLang [13] offers the feature of Robustness Analysis [6]. This is used to find a suitable selection of variables. Simply taken, the Robustness Analysis examines statistical dependencies between any input and output quantities the user desires. The data is obtained from a Monte Carlo simulation with small sample size. The input variables are simulated following statistical properties provided by the user, or they are varied systematically within given bounds. optiSLang comprises functions such as filters, fit tests or a principal component analysis of the correlation matrix, which shall reveal the most relevant influences on the output quantities observed. Results are plotted as coloured matrices, histograms etc. which make it easy to identify dependencies between variables. Two functions are explained in more detail in the following.

By computing the *quadratic correlation* it is tested, if one variable Y can be represented by a quadratic regression of another variable, X. The regression model is

$$\widehat{Y}(X) = a + bX + cX^2 \tag{12}$$

Samples of $\widehat{Y}$ are gained by inserting samples of X into eq. (12), values of Y itself are computed directly. Then the correlation coefficients $\rho_{Y\widehat{Y}}$ and $\rho_{\widehat{Y}Y} \neq \rho_{Y\widehat{Y}}$ are evaluated. The values range from 0 to 1, high values indicate a strong quadratic correlation between X and Y.

The *coefficient of determination* (CoD) is the ratio of variances of a regression model and the original variable. It indicates the amount of variability of an output variable Y, which can be explained by the variability of the input variable X underlying the regression model. For the quadratic regression of eq. (12):

$$R^2 = \frac{\sum_{i=1}^{n} \left(\widehat{Y}_i(X_i) - \mu_Y\right)^2}{\sum_{i=1}^{n} \left(Y_i - \mu_Y\right)^2} \tag{13}$$

Values of R^2 vary from 0 to 1. A high value, e.g. $R^2 = 0.8$, means that 80 % of the variability of Y can be explained by a quadratic relation between Y and X. However, this is no accuracy measure of the regression model. While the (quadratic) correlation coefficients only give information about the mutual relation of two variables, the CoDs allow for a comparison of the influences of all input variables to the output. Systematically omitting one variable at a time and observing the change of the CoD allows a ranking of the input variables by their influence on the result. In the example in sect. 5 it is shown, that for a random field representation, this ranking helps to reduce the number of random variables drastically.

4 Response Surface Methods

4.1 Formulation of Reliability

Safety and reliability analysis warrants serviceability and the exclusion of damage of a design during its life time. For numerical reasons, the reliablity is usually formulated by its complement, the probability of failure.

We can define any undesired or unsafe state of a response as the event $\mathcal{F}$ out of the set of all random variables $\mathbf{X}$, when the (inherently deterministic) state function $g(\mathbf{x})$ is less or equal to zero. The probability of occurence of this event, commonly probability of failure, is defined as

$$P(\mathcal{F}) = P[\{\mathbf{X} : g(\mathbf{x}) \leq 0\}] \tag{14}$$

and is computed as the integral over the failure domain

$$P(\mathcal{F}) = \underset{g(\mathbf{x})\leq 0}{\int \cdots \int} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{15}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of the basic random variables.

4.2 Global Polynomial Approximation

As already mentioned, an implicit formulation of the limit state, e.g. by nonlinear Finite Element analysis, is computationally expensive. Alternatively, the original state function $g(\mathbf{x})$ can be approximated by a response surface function $\tilde{g}(\mathbf{x})$ which has polynomial form [8, 9, 14, 15, 32, 33]. A commonly used method for response value approximation is the regression of a second order polynomial [3, 28]. Alternatively, the limit state $g(\mathbf{x}) = 0$ itself can be interpolated by second order polynomials [10, 31].

The global approximation schemes widely used in the application of RSM can be quite misleading due to the lack of information in certain regions of the random variable space. Standard second order polynomial approximations are not sufficiently flexible. Hence, the estimation of the failure probability using this global approximation may lead to large errors, in particular for small failure probabilities $P(\mathcal{F}) < 10^{-2}$ and a number of random parameters of $n > 5$. It is therefore required to avoid such undesirable approximation errors at reasonable computational effort.

4.3 Adaptive Response Surface Method

4.3.1 Moving Least Square approximation

A commonly used approximation method with minimal regression error within the support point values is the Moving Least Square approximation. The main advantage of this method is the flexibility for the approximation of highly nonlinear state and limit state functions. The proposed method is suitable for computing the reliability of complex models and is intended to provide reasonably accurate estimates of failure probabilities while maintaining computational efficiency.

Moving Least Square (MLS) functions can approximate locally clustered support point samples with higher local approximation quality. In addition, MLS improve the response surface model using additional support points. MLS is formulated as

$$\hat{y}(\mathbf{x}) = \sum_{i=1}^{n_b} h_i(\mathbf{x}) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x})\ \mathbf{a}(\mathbf{x}) \tag{16}$$

with a predefined number of base terms n_b, a vector of base functions $\mathbf{h}$ and the associated vector of the coefficients $\mathbf{a}$. [23] formulate a local MLS approximation as

$$\hat{y}(\mathbf{x}, \mathbf{x}_j) = \sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}_j)\ \mathbf{a}(\mathbf{x})$$

with $j = 1, ..., n_s$ support points. The approximate coefficient vector $\mathbf{a}$ can be calcu-

lated using the weighted least square postulate

$$\begin{aligned} S(\mathbf{x}) &= \sum_{j=1}^{n_s} w(\mathbf{x}-\mathbf{x}_j)\left(\hat{y}(\mathbf{x},\mathbf{x}_j)-y(\mathbf{x}_j)\right)^2 \\ &= \sum_{j=1}^{n_s} w(\mathbf{x}-\mathbf{x}_j)\left(\sum_{i=1}^{n_b} h_i(\mathbf{x}_j)a_i(\mathbf{x})-y(\mathbf{x}_j)\right)^2 \\ &= (\mathbf{Ha}-\mathbf{g})^T\mathbf{W}(\mathbf{x})(\mathbf{Ha}-\mathbf{g}) \rightarrow \min \end{aligned} \tag{17}$$

with the weighting function $w(\mathbf{x}-\mathbf{x}_j)$ and

$$\begin{aligned} \mathbf{g} &= [y(\mathbf{x}_1) \;\; y(\mathbf{x}_2) \;\; ... \;\; y(\mathbf{x}_{n_s})]^T \\ \mathbf{H} &= [\mathbf{h}(\mathbf{x}_1) \;\; \mathbf{h}(\mathbf{x}_2) \;\; ... \;\; \mathbf{h}(\mathbf{x}_{n_s})]^T \\ \mathbf{h}(\mathbf{x}_j) &= [h_1(\mathbf{x}_j) \;\; h_2(\mathbf{x}_j) \;\; ... \;\; h_{n_b}(\mathbf{x}_j)]^T \\ \mathbf{W}(\mathbf{x}) &= \text{diag}[w(\mathbf{x}-\mathbf{x}_1) \;\; w(\mathbf{x}-\mathbf{x}_2) \;\; ... \;\; w(\mathbf{x}-\mathbf{x}_{n_s})] \end{aligned}$$

The least square error $S(\mathbf{x})$ may be a minimum in the case that the partial gradients are zero.

$$\frac{\partial S(\mathbf{x})}{\partial \mathbf{a}} = 0$$

So using the equation (17), a linear equation system gives an estimation of the coefficient vector $\mathbf{a}$

$$\mathbf{a}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x})\,\mathbf{B}(\mathbf{x})\,\mathbf{g} \tag{18}$$

with

$$\begin{aligned} \mathbf{M}(\mathbf{x}) &= \mathbf{H}^T\,\mathbf{W}(\mathbf{x})\,\mathbf{H} \\ \mathbf{B}(\mathbf{x}) &= \mathbf{H}^T\,\mathbf{W}(\mathbf{x}) \end{aligned}$$

Since the matrix of the base function $\mathbf{M}(\mathbf{x})$ should be non-singular, always a sufficient number of n_s immediately neighboured support points has to be available. The number must be at least as large as number of the base terms. Equation (18) inserted in (16) gives the approximation function

$$\hat{y}(\mathbf{x}) = \mathbf{h}^T(\mathbf{x})\,\mathbf{M}^{-1}(\mathbf{x})\,\mathbf{B}(\mathbf{x})\,\mathbf{g}$$

An approximation quality as accurate as possible requires a weighting function which is larger than zero $w(\mathbf{x}-\mathbf{x}_j) > 0$ and monotonically decreasing $w(\|\mathbf{x}-\mathbf{x}_j\|)$ inside of a small sub space $\Omega_s \subset \Omega$. So the influence of supports far from the actual coordinates is unimportant. A uniform weighting is given by a symmetry condition $w(\mathbf{x}-\mathbf{x}_j) = w(\mathbf{x}_j-\mathbf{x}) = w(\|\mathbf{x}-\mathbf{x}_j\|)$. Usually, an exponential function is used in this way:

$$w(\|\mathbf{x}-\mathbf{x}_j\|) = \begin{cases} e^{-\left(\frac{\|\mathbf{x}-\mathbf{x}_j\|}{D\alpha}\right)^2} & \|\mathbf{x}-\mathbf{x}_j\| \le D \\ 0 & \|\mathbf{x}-\mathbf{x}_j\| > D \end{cases} \tag{19}$$

with a constant

$$\alpha = \frac{1}{\sqrt{-\,log\,0.001}}$$

and a influence radius D to be chosen. It is obvious that the smaller D the better the response values of the support points fit the given values. But as mentioned above at least n_b support points have to be available in every point to be approximated. Therefore it is possible that a D has to be chosen which leads to a large shape function error at the support points.

To avoid these problems a new regularized weighting function was introduced by Most and Bucher [27]:

$$w_R(\|\mathbf{x}-\mathbf{x}_j\|) = \begin{cases} \dfrac{\hat{w}_R(\|\mathbf{x}-\mathbf{x}_j\|)}{\sum_{i=1}^{n_s} \hat{w}_R(\|\mathbf{x}-\mathbf{x}_i\|)} & \|\mathbf{x}-\mathbf{x}_j\| \le D \\ 0 & \|\mathbf{x}-\mathbf{x}_j\| > D \end{cases} \tag{20}$$

with

$$\hat{w}_R(d) = \frac{\left(\left(\dfrac{d}{D}\right)^2 + \varepsilon\right)^{-2} - (1+\varepsilon)^{-2}}{(\varepsilon)^{-2} - (1+\varepsilon)^{-2}}\ ;\ \varepsilon \ll 1 \tag{21}$$

It is recommended by Most and Bucher to use the value

$$\varepsilon = 10^{-5}$$

This new regularized weighting function works better than the exponential function. But if the ratio of the minimal distance among the supports to the extent of areas where are no supports becomes worse the same problems occurs again. As a matter of fact a large D is needed to approximate for coordinates where no support points are around and a small D is needed for coordinates where are a lot of support points in order to reach a minimal approximation error. To comply with these conditions it is necessary to use a function $d(\mathbf{x})$ for the influence radius instead of a constant D.

4.3.2 Adaptive design of experiment

In particular, these response surfaces can be adaptively refined to consistently increase the accuracy of the estimated failure probability. This is especially suitable for the reliability analysis of complex nonlinear structures. An arbitrary number of check points even in high local concentration can be used without approximation problems. Using a deterministic design of experiments, the necessary number of support points becomes very high with an increasing number of random variables.

To decrease the number of support points in an optimized way, the so called D-optimality criterion is used. A discussion of this criterion is presented by Box and

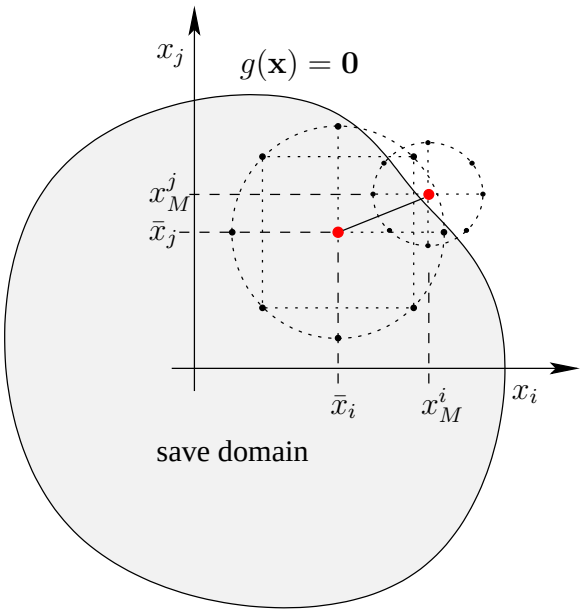


Figure 1: Adaptive design of experiment in the random space.

Draper [4]. The effectiveness of a design in satisfying the minimum variance (D-Optimal) criterion is expressed by the D-efficiency of the design. Myers and Montgomery [29] provide more exact specifications of the D-Optimal criteria and further criteria called alphabetic optimality criteria are described. However, the design of experiments in the first iteration should explore the random space including safe and unsafe domain as accurate as possible. An approach to this is given by Klingmüller and Bourgund [22] with

$$x_i = \bar{x}_i \pm f\sigma_{x_i}$$

whereby

$$f = \Phi^{-1}(P(\mathcal{F})) = 3, \ldots, 5$$

is a factor depending on the assumed failure probability. Bucher and Bourgund [8] give an efficient possibility to adaption a design of experiment in the next iterations with

$$\mathbf{x}_M = \bar{\mathbf{x}} + (\mathbf{x}_D - \bar{\mathbf{x}})\frac{g(\bar{\mathbf{x}})}{g(\bar{\mathbf{x}}) - g(\mathbf{x}_D)}$$

with

$$\mathbf{x}_D = E[\mathbf{X}|g(\mathbf{x}) \leq 0]$$

as shown in Figure 4.3.1. This is achieved by a combination of random search strategies (based on the adaptive sampling approach, see [7] for details) as well as deterministic search refinement. In such a way 3 to 6 iteration steps are necessary for a sufficient convergence in most practical applications. This adaptive design of experiment using a D-optimal linear or quadratic design in combination with the improved Moving Least Square approximation is suitable up to $n \leq 20$ random parameters.

5 Example: Cylindrical Shell

5.1 Structure and Random Field Properties

The reliablity of a geometrically imperfect shell with properties as given in fig. 2 is analysed. The cylinder has a Navier–type suppport along the top and bottom edges and is loaded along the top edge with a continuous vertical unit load. The structure is modelled with a 9-node isoparametric shell finite element type within the program SLang [21].

A random field is applied on the structure in order to model geometrical imperfections. The random properties are coordinate deviations from the perfect structure in the cylinder's radial direction. Thus the random field is discretized at the nodes of the finite element mesh. It has zero mean and a standard deviation of $\sigma_H = 10^{-3}$ mm, which is roughly a hundredth of the radius. The orthogonal field has different correlation functions along the perimeter and height as plotted in fig. 3. The spatial correlation structure with respect to one node on the bottom edge is visualized, too.

The reliability of the imperfect structure towards stability failure shall be studied here. The method of analysis to be applied has to be a compromise between accuracy and computing time. For the example observed here, which shows a pre–buckling behaviour close to linear and a sudden buckling failure, the linear buckling analysis suffices. It is very fast and not prone to being trapped on a postbuckling equilibrium state.

The limit load for reliability analysis is adopted from the buckling load of the perfect structure subtracted a "safety margin". Two case studies at different safetey levels have been carried out with limit state functions as follow:

$$g(\mathbf{X}) = \begin{cases} F_{buckling} - 34 \text{ kN} \leq 0 \ , \ \text{case 1} \\ F_{buckling} - 30 \text{ kN} \leq 0 \ , \ \text{case 2} \end{cases} \tag{22}$$

Failure is defined as the event, that the limit state function takes values of less than zero.

5.2 Preliminary studies

A Robustness Analysis is performed with optiSLang, wherein all 396 variables of the random field are involved and, among others, their relations to the critical buckling loads of the simulated imperfect structures are examined.

No significant linear correlations could be found. Instead, strong quadratic correlations are observed between the first 14 random variables and the critical load (where "first" indicates those variables with the highest variances, i.e. highest eigenvalues after decomposition of the covariance matrix, cf. sect. 2.2). For variables of order higher than 14, the quadratic correlation coefficients are close to zero. The quadratic correlation matrix as explained in section 3 is displayed in fig. 4. The nonlinear dependency becomes obvious by a scatter plot, in fig. 5, e.g., of input variable no. 1 vs.

Wall thickn. [mm]	0.197
Radius [mm]	101.6
Height [mm]	139.7
Young's mod. [N/mm^2]	$6.895 \cdot 10^4$

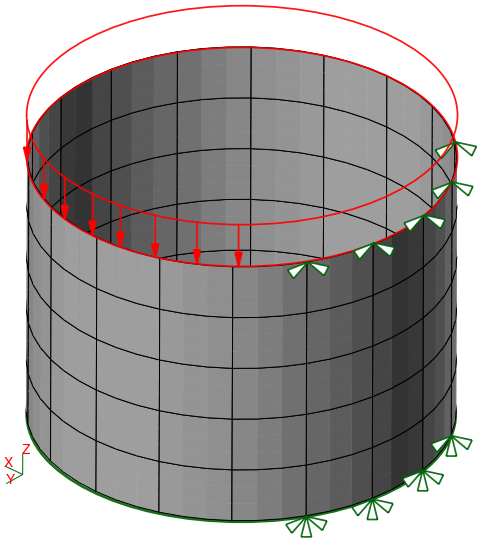
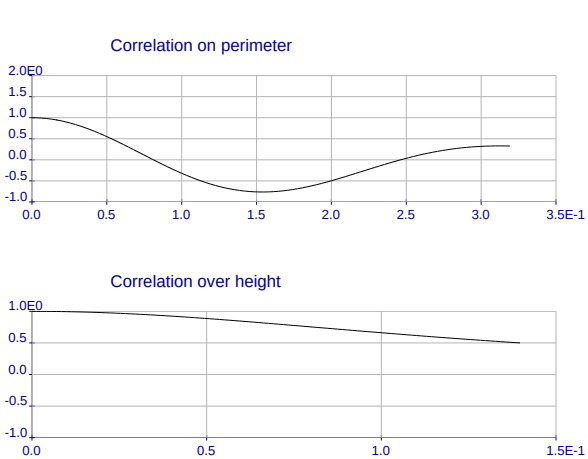


Figure 2: Finite element model of the cylindrical shell with schematic loads and supports.



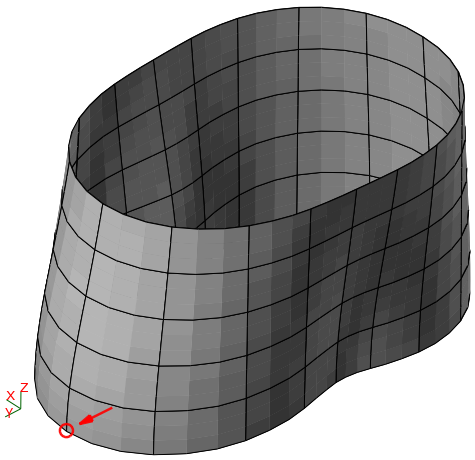


Figure 3: Correlation functions over perimeter (above) and height (below) and spatial correlation structure with respect to the marked node (right).

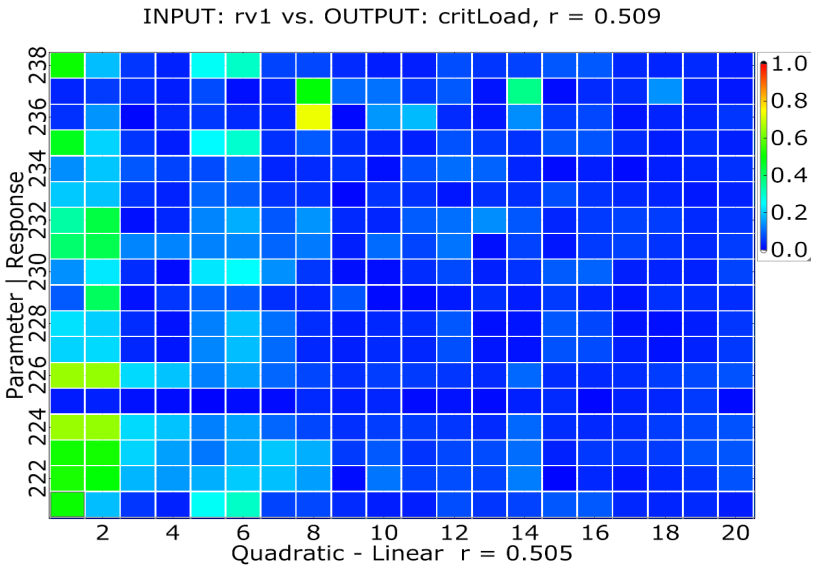


Figure 4: Matrix (part) of quadratic correlation coefficients. The lowest row shows correlations of the critical load with the first 20 random variables.

the critcal load.

Based on the quadratic regression of eq. (12) for the buckling load, with each random variable set in for X successively, the coefficients of determination, eq. (13), are computed. The sum of all values is less than 100 %. That means, the variance of the critical load cannot be fully explained by a quadratic relation to the input variables. The results are sorted and plotted as bar diagram, fig. 6. The strongest influences can easily be identified. A closer look reveals that not all of the "first 14" variables (see above) are most relevant, but a few variables assigned to higher order eigenvectors of the covariance matrix as well. The eigenvalues and eigenvectors of the covariance matrix (used as random amplitudes and shape functions in the reliablity analysis) which are selected by the criterion of the "top 14" coefficients of determination are that of order 1, 2, 5, 6, 15, 21, 22, 26, 29, 30, 32, 34, 83 and 197.

5.3 Reliability Analysis – Case 1

For the first limit state condition as given in equation 22, the reliability of the structure is studied by means of a "plain" Monte Carlo simulation [as in 1], with the limit state function as defined by eq. (22). Three variants are computed: as a reference, the full representation of the random field, which employs 396 random variables, is used. Second, the "first 14" variables were selected with the criterion defined by eq. 11 and third, a set of random variables with the "top 14" coefficients of determination, cf. sect. 5.2. In each case, a sample with 36000 realizations is generated by Latin Hypercube Sampling [16, 20, 25]. No other variance reduction scheme such as Importance Sampling is applied. Because the random field defines the structural geometry and hence the structural behaviour, the limit state function cannot be programmed explicitely, but a

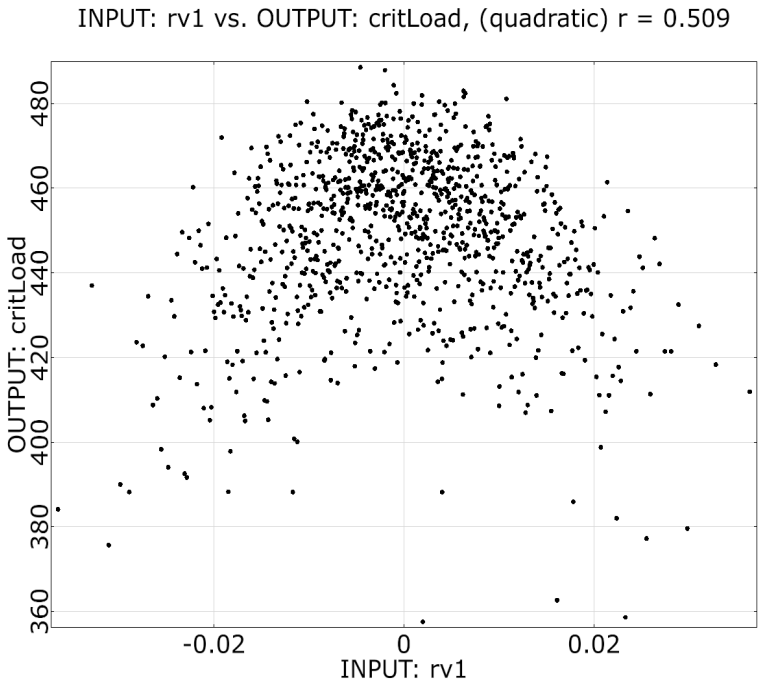


Figure 5: Anthill plot of critical load vs. first random variable.

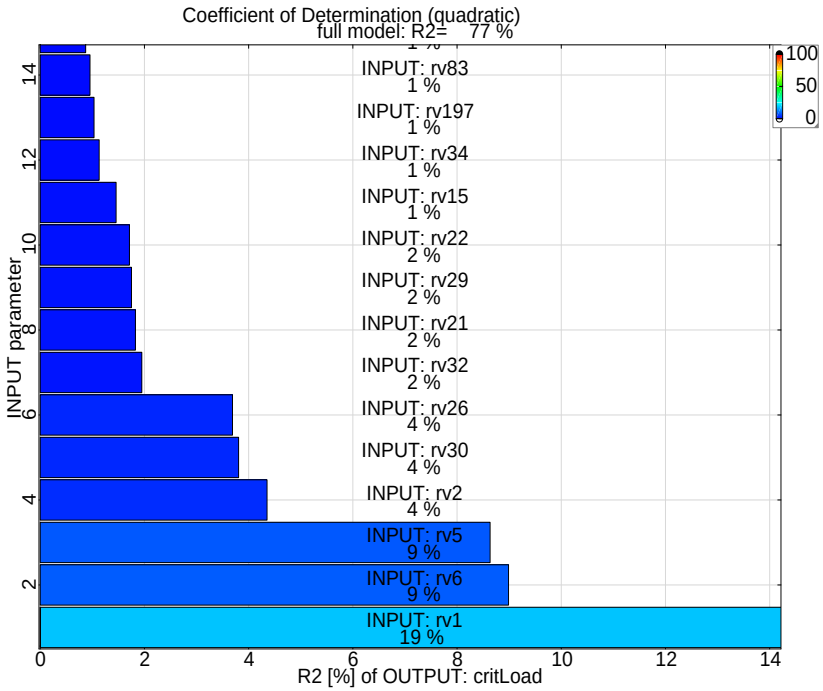


Figure 6: The top 14 coefficients of determination of critical load, quadratic regression model for random input variables.

linear buckling analysis has to be carried out for each sample.

The failure probabilities computed with the different sets of variables are listed in table 5.3. The so–called statistical error, i.e. the standard deviation of the estimator of the failure probability, is listed as well, in order to assess the confidence in the result. The simulation results with all variables and the "top 14" selection show a good quality. With the "first 14" set of variables, the probability of failure is underestimated by more than a magnitude. The statistical error in this case in unacceptably high, despite the huge sample size. This set of random variables is able to represent the random field well, but is not able to model the structural behaviour. The result obtained with the "top 14" selection is close to the reference, although it tends to be lower, too. Obviously, this selection criterion provides a good compromise for both modelling the stochastic and the mechanical problem.

No. of random variables Selection criterion	396 none (all)	14 highest $\sigma^2_{X_i}$ ("first 14")	14 highest R^2 ("top 14")
Prob. of failure P_f	$9.7 \cdot 10^{-3}$	$2.8 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$
Statistical error σ_{P_f} $\mathrm{cov}(P_f)$	$5.2 \cdot 10^{-4}$ 5 %	$8.8 \cdot 10^{-5}$ 32 %	$3.2 \cdot 10^{-4}$ 9 %

Table 1: Probabilities of failure for different sets of random variables.

5.4 Reliability Analysis – Case 2

For the second limit state condition of eq. 22, the reference result is computed by directional sampling with 3000 samples and in total $N = 15690$ finite element solver evaluations (see figure 8) to give an estimated failure probability of $1.966 \cdot 10^{-5}$ and a standard deviation of the estimator of $3.927 \cdot 10^{-6}$. The history of the failure probability and the standard deviation of the estimator are shown in fig. 7.

Applying the new adaptive response surface method to this state function, as shown in Figures 9 and 10, leads to an accurate estimation of the failure probability already after the first adaptation. ARSM starts with an initial D-optimal quadratic design of experiment with an initial axial multiplier of 2.0 and 4000 samples for each adaptive simulation on the surrogate model. For the second design of experiment only a D-optimal linear scheme is used. So in total only $N = 229$ finite element evaluations are necessary to estimate the failure probability of $1.392 \cdot 10^{-5}$.

6 Conclusion

In the present article, a step-by-step procedure for calculating the reliability of a structural system with random imperfections was introduced and validated. Such a com-

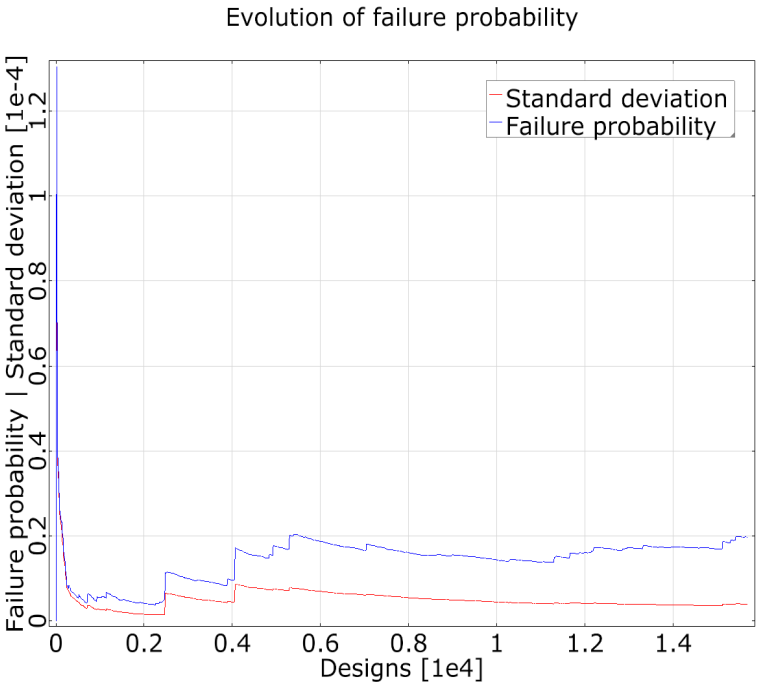


Figure 7: History of the failure probability and the standard deviation of the estimator – Directional Sampling.

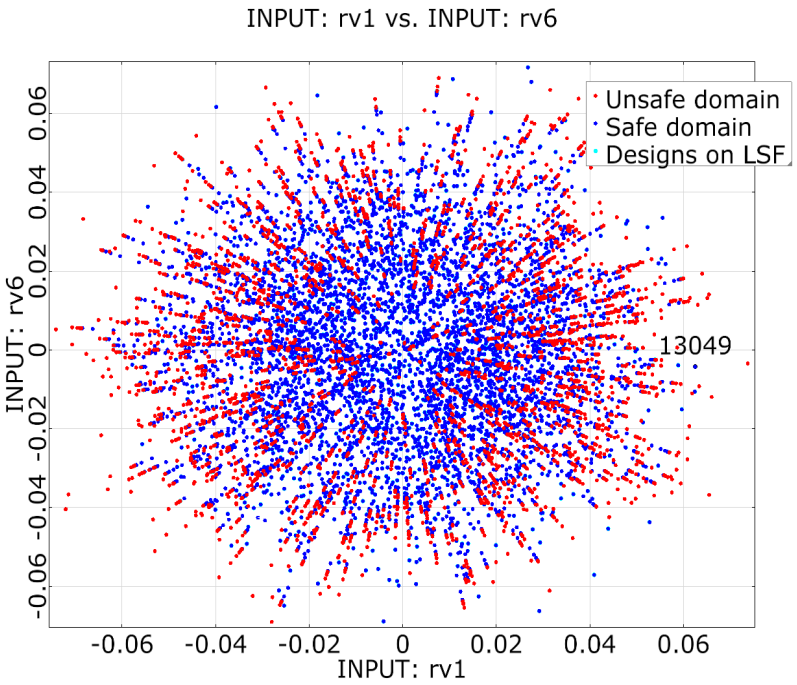


Figure 8: Two-dimensional anthill plot of the $N = 3000$ simulated directions in the subspace of the first and second random variable.

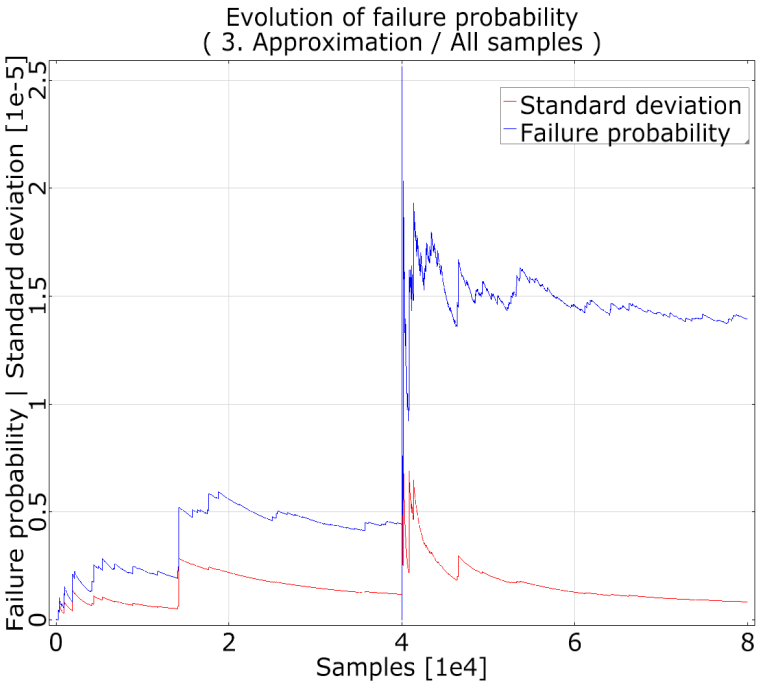


Figure 9: History of the failure probability and the standard deviation of the estimator – ARSM.

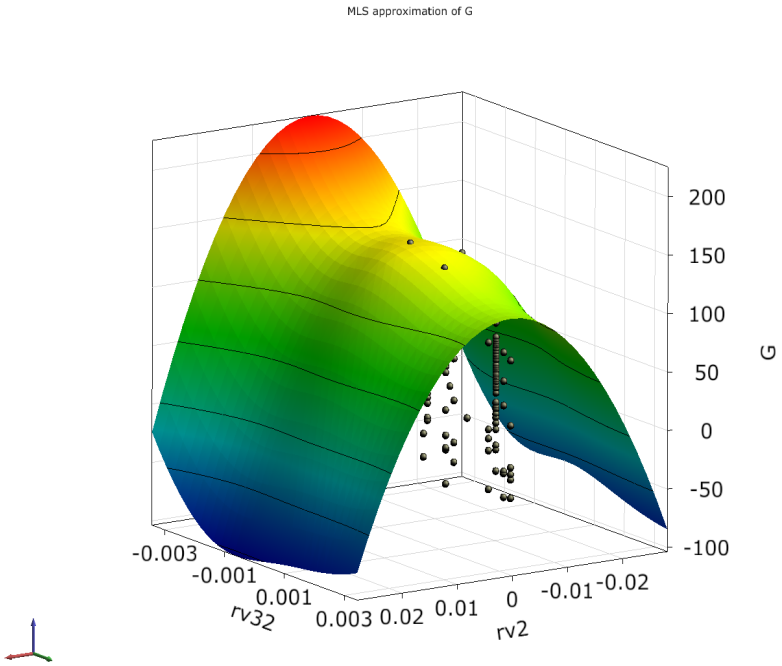


Figure 10: Moving Least Square approximation of the state function $g(x)$ in the subspace of variables 2 and 32.

plex problem still needs preliminary studies, but is already applicable to realistic problems. Within such preparations, attention must be paid on the analysis of the mechanical problem and choice of the calculation method. A compromise between accuracy and computing time should be found.

The reliability was computed by advanced Monte Carlo methods, with the imperfections modelled as random fields. This may require a huge number of random variables, which in turn may cause numerical difficulties in the computation of the probability of failure. Hence the first step of the analysis in the preparation phase is a suitable selection of the random variables. Here, a Robustness Analysis is performed for this purpose, which requires relatively few additional samples compared to the sample size needed for the reliability analysis itself. The Robustness Analysis comprises, among other functions, a quadratic regression of the state function by the input random variables. It turned out that for the underlying geometrically non–linear problem, this approach helps to detect the important input variables.

A new Adaptive Response Surface Method is introduced that allows for a reliability analysis with high accuracy and efficiency. The surrogate model is based on an improved Moving Least Square approximation combined with an adaptive design of experiments. The main advantage of this method is its flexibility for the approximation of highly nonlinear limit state functions. The response surface model is improved by successively adding new support points leading to a local refinement of the model. Moving Least Square functions are able to handle locally clustered support points to provide an increased local approximation quality. However, the design of experiments in the first iteration should explore the random space including both the safe and unsafe domain as accurate as possible. This is achieved by a combination of random search strategies (based on the adaptive sampling approach) as well as deterministic search refinement.

In order to obtain a fast simulation procedure on the response surface, an Adaptive Importance Sampling concept is used. The proposed method is very robust and efficient for every safety level up to $n \leq 20$ random parameters.

The kind of reliability analysis presented here is called non–parametric, because it depends on stochastic properties defined continuously on the entire structure, but not on geometry parameters. Quite often the randomness of the geometry is introduced by few random structural parameters, such as radius, material thickness etc. These values can be generated by CAD programs. In fact, only variants of the perfect structure are generated that way. It is impossible to consider the random fluctuations over space, which result e.g. from manufacturung tolerances. The proposed approach is more effective for this class of problems.

References

[1] V. Bayer. *Zur Zuverlässigkeitsbeurteilung von Baukonstruktionen unter dynamischen Einwirkungen*. Institut für Strukturmechanik, Bericht 3/99. Bauhaus

- Universität Weimar, Weimar, 1999.

[2] P. Bjerager. Probability integration by directional simulation. *Journal of Engineering Mechanics, ASCE*, Vol. 114(No. 8):1285 – 1302, 1988.

[3] G. E. P. Box and N. R. Draper. *Empirical Model Building and Response Surfaces*. John Wiley and Sons, New York,USA, 1987.

[4] M. J. Box and N. R. Draper. Factorial designs, the $|\mathbf{x}^t\mathbf{x}|$ criterion, and some related matters. *Technometrics*, Vol. 13(4):731 – 742, 1971.

[5] C. E. Brenner. *Ein Beitrag zur Zuverlässigkeitsanalyse von Strukturen unter Berücksichtigung von Systemuntersuchungen mit Hilfe der Methode der Stochastischen Finite Elemente*. Thesis (Dr. tech.). Leopold Franzens Universität Innsbruck, Innsbruck, 1995.

[6] C. Bucher. Robustness analysis in structural optimization. *Structure and Infrastructure Engineering*, 2007. to appear.

[7] C. G. Bucher. *Adaptive Sampling - An Iterative Fast Monte Carlo Procedure*. Structural Safety, vol. 5, no. 2 edition, 1988.

[8] C. G. Bucher and U. Bourgund. *Efficient Use of Response Surface Methods*. Bericht Nr. 9 - 87, Institut für Mechanik, Universitët Innsbruck, Innsbruck,Austria, 1987.

[9] C. G. Bucher and U. Bourgund. A fast and efficient response surface approach for structural reliability problems. *Structural Safety*, 7(1):57–66, 1990.

[10] C. G. Bucher, Y. M. Chen, and G. I. Schuëller. Time variant reliability analysis utilizing response surface approach. In P. Thoft-Christensen, editor, *Proc., 2nd IFIP Working Conference on Reliability and Optimization of Structural Systems*, pages 1 – 14. Springer Verlag, Berlin,Germany, 1988.

[11] A. Der Kiureghian and J.-B. Ke. The stochastic finite element method in structural reliability. *Probabilistic Engineering Mechanics*, 3(2):135–141, 1988.

[12] O. Ditlevsen and P. Bjerager. Plastic reliability analysis by directional simulation. *Journal of Engineering Mechanics, ASCE*, Vol. 115(No. 6):1347 – 1362, 1989.

[13] DYNARDO GmbH. *optiSLang – the optimizing Structural Language, Version 2.1*. Weimar, 2006.

[14] S. Engelund and R. Rackwitz. Experiences with experimental design schemes for failure surface estimation and reliability. In Y. K. Lin, editor, *ASCE Specialty Conference on Probabilistic Mechanics and Structural and Geotechnical Reliability*, pages 244 – 247. Proceedings, 6th ASCE, New York,USA, 1992.

[15] L. Faravelli. *Response Surface Approach for Reliability Analysis*. Pubblicazione n. 160, Dipartimento di Meccanica Strutturale Dell' Universita di Pavia, PaviaItaly, 1986.

[16] A. Florian. An efficient sampling scheme: Updated latin hypercube sampling. *Probabilistic Engineering Mechanics*, 7:123–130, 1992.

[17] R. Ghanem and P.D. Spanos. *Stochastic Finite Elements - a Spectral Approach*. Springer, New York, Berlin, 1991.

[18] J. M. Hammersley and D. C. Handscomb. *Monte Carlo Methods*. Methuen & Co. Ltd., London, 1967.

[19] T. Hisada and S. Nakagiri. Stochastic finite element method developed for structural safety and reliability. In *Third Int. Conf. on Structural Safety and Reliability*, pages 409–417, Trondheim, 1981.

[20] D. E. Huntington and C. Lyrintzis. Improvements to and limitations of latin hypercube sampling. *Probabilistic Engineering Mechanics*, 13(4):245–253, 1998.

[21] Institut für Strukturmechanik. *SLang – The Structural Language, Users Manual*. Weimar.

[22] O. Klingmüller and U. Bourgund. *Sicherheit und Risiko im Konstruktiven Ingenieurbau*. Vieweg, Braunschweig, Wiesbaden, Germany, 1992.

[23] P. Lancaster and K. Salkauskas. *Curve and surface fitting; an introduction*. Academic Press, London, 1986.

[24] P.-L. Liu and A. Der Kiureghian. Multivariate distribution models with prescribed marginals and covariances. *Probabilistic Engineering Mechanics*, 1(2): 105–112, 1986.

[25] M. D. McKay, W. J. Conover, and R. J Beckmann. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21:239–245, 1979.

[26] R. E. Melchers. *Radial importance sampling for structural reliability*. Engrg. Mech., October, asce, 116(1) edition, 1990.

[27] T. Most and C. G. Bucher. A moving least squares weighting function for the element-free galerkin method which almost fulfills essential boundary conditions. *Structural Engineering and Mechanics*, 21(3):315 – 332, 2005.

[28] R. H. Myers. *Response Surface Methodology*. Allyn and Bacon Inc., BostonUSA, 1971.

[29] R. H. Myers and D. C. Montgomery. *Response Surface Methodology - Process and Product Optimization Using Designed Experiments*. John Wiley & Sons, Inc., New York, 1995.

[30] A. Nataf. *Détermination des distributions de probabilités dont les marges sont données*, volume 225 of *Comptes Rendus de l'Académie des Sciences*. 1962.

[31] W. Ouypornprasert and C. G. Bucher. An efficient scheme to determine reponse functions for reliability analyses. *Universität Innsbruck, Internal Working Report No. 30*, 30:1 – 39, 1988.

[32] R. Rackwitz. *Response Surfaces in Structural Reliability*. Berichte zur Zuverlëssigkeitstheorie der Bauwerke, Heft 67/1982, Laboratorium für den konstruktiven Ingenieurbau, Technische Universitët München, MünchenGermany, 1982.

[33] M. R. Rajashekhar and B. R. Ellingwod. *A new look at the response surface approach fo reliability analysis*. Structural Safety, vol. 12, no. 1 edition, 1993.

[34] R. Y. Rubinstein. *Simulation and the Monte Carlo Method*. John Wiley & Sons, New York, 1981.

[35] Y. Schorling. *Beitrag zur Stabilitätsuntersuchung von Strukturen mit räumlich korrelierten geometrischen Imperfektionen*. Institut für Strukturmechanik, Bericht 2/98. Bauhaus - Universität Weimar, Weimar, 1998.