

## Lectures

# Design Reliability Analysis

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**Summary** A large number of problems in manufacturing processes, production planning, finance and engineering design require an understanding of potential sources of variations and quantification of the effect of variations on product behavior and performance. Traditionally, in engineering problems uncertainties have been formulated only through coarse safety factors. Such methods often lead to overdesigned products.

First, this paper gives an overview over the state of the art in reliability analysis methods to describe model uncertainties and to calculate reliability and safety. Different methods exist, but they all have a limited area of application. The aim of the first part is to investigate the applicability of certain methods for specific problems and to give a general indication, which method is appropriate for certain problem classes, implemented in **optiSLang**.

In addition, a new adaptive response surface method is introduced to analyse the design reliability with high accuracy and efficiency. Whereby the surrogate model is based on an improved moving least square approximation combined with an adaptive design of experiment. In order to obtain a fast simulation procedure on the response surface an adaptive importance sampling concept is developed.

Several numerical examples show the applicability of this concept for highly nonlinear state and limit state functions and multiple design points and separated unsafe domains. Furthermore, within a structural reliability analysis example a discretization of random fields in combination with a robustness evaluation to identify the relevant random variables is introduced. The so-called nonparametric structural reliability analysis is a new method to estimate the safety and reliability of finite element structures in cases where a CAD-based parametrization is not possible or not meaningful. The probabilistic and structural analysis tasks are performed with the **optiSLang**, **SoS** and **SLang** software packages.

**Keywords** reliability analysis, random fields, robustness evaluation, adaptive response surfaces, moving least square approximation, adaptive design of experiment, adaptive importance sampling, latin hypercube sampling

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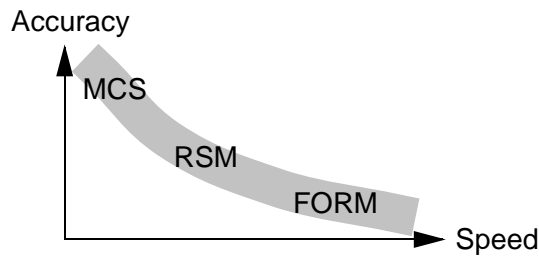


Figure 1: Performance of methods for stochastic analysis. a.) Monte Carlo simulation (MCS), b.) Response Surface Method (RSM), c.) First Order Reliability Method (FORM).

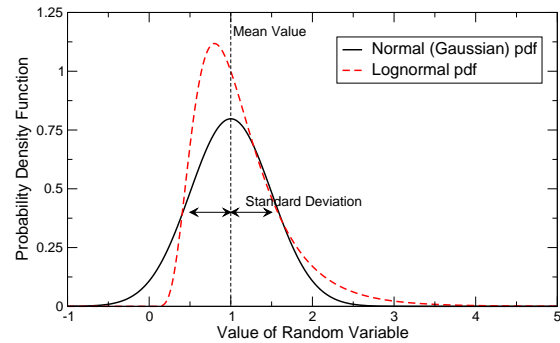


Figure 2: Probability density function  $f_X(x)$  of the normal and lognormal distribution with mean and standard deviation.

## 1 Introduction

Within many engineering fields, structural design must meet challenging demands of cost-effectiveness. This leads to light-weight, highly flexible, and consequently vulnerable structures. In order to assess potential risks associated with such a design, the structural analysis must take into account available information on the randomness of loads and structural properties. It means that the analysis should include reliability estimates in an appropriate manner.

Considering the properties of the computational analysis realistically it is necessary to take into account some uncertainty. This uncertainty can be conveniently described in terms of probability measures, such as distribution functions. It is a major goal of stochastic computational analysis to relate the uncertainties of the input variables to the uncertainty of the model response performance. The stochastic models for these uncertain parameters can be one of the following:

- Time-independent random problems (Reliability analysis)
  - Random variables (constant in time and space)
  - Random fields (correlated fluctuations in space)
- Time-dependent random problems (First passage reliability analysis)
  - Random processes (correlated fluctuations in time)

Within a computer implementation it becomes necessary to discretize random fields/processes in time and space. This immediately leads to a considerable number of random variables.

The aims of stochastic analysis include safety measures such as time-invariant failure probabilities or safety indices, response variability in terms of standard deviations or time-variant first passage probabilities. The available solution methods for these tasks are exact methods (numerical integration, Monte Carlo simulation – MCS), approximate (first and second order reliability method – FORM/SORM, response surface method - RSM) or other techniques (e.g. based on Markov vector theory in stochastic dynamics).

The stochastic analysis software **SLang** (– the Structural Language) includes several methods solving all off these stochastic models. Currently, **optiSLang** (– the optimizing Structural Language) supports methods to analyse random variables only. In addition, the **SoS** (– Statistics on Structures) add-on tool to **optiSLang** provides methods to solve random fields.

Although Monte Carlo methods are most versatile, intuitively clear, and well understood, the computational cost (the number of computational runs required) in many cases is prohibitive. Thus approximations become important, which can be based e.g. on the response surface methods (RSM) or first/second order reliability methods (FORM/SORM). For the feasibility of response surface approaches it is quite essential to reduce the number of variables to a tractable amount. This may require extensive sensitivity analysis in order to identify the relevant random variables. This is particularly important for random

Property	SD/Mean %
Metallic materials, yield	15
Carbon fiber composites, rupture	17
Metallic shells, buckling strength	14
Junction by screws, rivet, welding	8
Bond insert, axial load	12
Honeycomb, tension	16
Honeycomb, shear, compression	10
Honeycomb, face wrinkling	8
Launch vehicle , thrust	5
Transient loads	50
Thermal loads	7.5
Deployment shock	10
Acoustic loads	40
Vibration loads	20

Table 1: Sources of uncertainties (Klein *et al.* (1994)) given by standard deviation (SD) and mean value as shown in Figure 2.

variables arising from discretization of random fields or processes. In this context, close coupling between the tools for stochastic and computational analyses is essential. Simplifications can be based on the following items

- Global variance-based sensitivity or robustness analysis of the structural response with respect to the random variables. Again, this aims at reducing the number of random variables needed.
- Concentrate random sampling in the region which contributes most to the total failure probability. This is generally called “importance sampling”. It is important to note that most importance sampling strategies work best with a low number of random variables.
- Approximation of the numerical response by a class of simple mathematical functions. This is the so-called “response surface method”. Again, it is vital that the number of random variables be kept small.

As a very simplistic rule-of-the-thumb, Fig. 1 gives the accuracy/speed ratio for some solution methods as mentioned above. However, there are situations in which MCS can be comparably fast or FORM can be comparably accurate.

Probabilistic analysis typically involves two areas of statistical variability as shown in Table 1. The first group consists of the uncontrollable uncertainties and tolerances. These include material property variability, manufacturing process limitations, environmental variability, such as temperature, operating processes (misuse) and result scatter arising from deterioration. The second group – the controllable parameters – involves design configurations, geometry, loads, constraints and manufacturing process settings.

## 2 Reliability Analysis

### 2.1 Introduction

Numerical methods, e.g. for structural analysis have been developed quite substantially over the last decades. In particular, finite element methods and closely related approximations became the state of the art. The modeling capabilities and the solution possibilities lead to an increasing refinement allowing for more and more details to be captured in the analysis. On the other hand, however, the need for more precise input data became urgent in order to avoid or reduce possible modeling errors. Such errors could eventually render the entire analysis procedure useless. Typically, not all uncertainties encountered in structural analysis can be reduced by careful modeling since their source lies in the intrinsic randomness



of natural phenomena. It is therefore appropriate to utilize methods based on probability theory to assess such uncertainties and to quantify their effect on the outcome of structural analysis.

The following chapter presents an overview of probability-based methods to describe model uncertainties and to calculate reliability and safety. Different methods exist, but they all have a limited area of application. The aim of this chapter is to investigate the applicability of certain methods for specific problems and to give a general indication, which method is appropriate for certain problem classes, implemented in **optiSLang**.

## 2.2 Formula of the Failure Probability

Safety and reliability analysis warrants the exclusion of damage and design collapse during the life time. Probability of surviving is the numerical quantity of safety and reliability and the probability of failure is the complement.

We can define any undesired or unsafe state of a response as an event  $\mathcal{F}$  out of the set of all random variables  $\mathbf{X}$  such a way that the assign state function  $g(\mathbf{x})$  is less or equal to zero. Generally, failure (i.e. an undesired or unsafe state of the response) is defined in terms of a limit state function  $g(\cdot)$ , i.e. by the set  $\mathcal{F} = \{\mathbf{X} : g(\mathbf{X}) \leq 0\}$ . Frequently,  $Z = g(\mathbf{X})$  is called *safety margin*. As indicated in Fig. 3, the

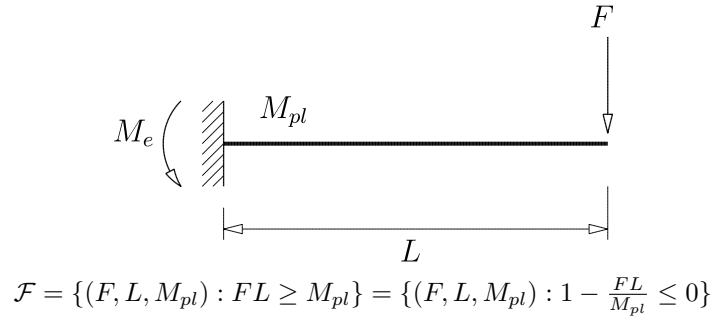


Figure 3: Structural system and several unique failure conditions.

definition of the limit state function is not unique. The failure probability is defined as the probability of the occurrence of the unsafe event  $\mathcal{F}$ :

$$P(\mathcal{F}) = P[\{\mathbf{X} : g(\mathbf{X}) \leq 0\}] \quad (1)$$

This quantity is *unique*, i.e. not depending on the particular choice of the limit state function. The response behavior near the failure state is most important in the reliability analysis. The random design parameters, such as loadings, material parameters and geometry, are the set of basic random variables  $\mathbf{X}$  which determine the probabilistic response of numerical model. The failure condition is defined by a deterministic limit state function

$$g(\mathbf{x}) = g(x_1, x_2, \dots, x_n) \leq 0$$

as shown in Fig. 2.2. The failure probability of a design is given by

$$P(\mathcal{F}) = P[\mathbf{X} : g(\mathbf{X}) \leq 0] = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (2)$$

where  $f_{\mathbf{X}}(\mathbf{x})$  is the joint probability density function of the basic random variables.

## 2.3 FORM - First Order Reliability Method

The FORM-Concept is based on a description of the reliability problem in standard Gaussian space. Hence transformations from correlated non-Gaussian variables  $\mathbf{X}$  to uncorrelated Gaussian variables  $\mathbf{U}$  with zero mean and unit variance are required. This step is called Rosenblatt-transformation. Then a linearization is performed in  $\mathbf{u}$ -space. The expansion point  $\mathbf{u}^*$  is chosen such as to maximize the PDF

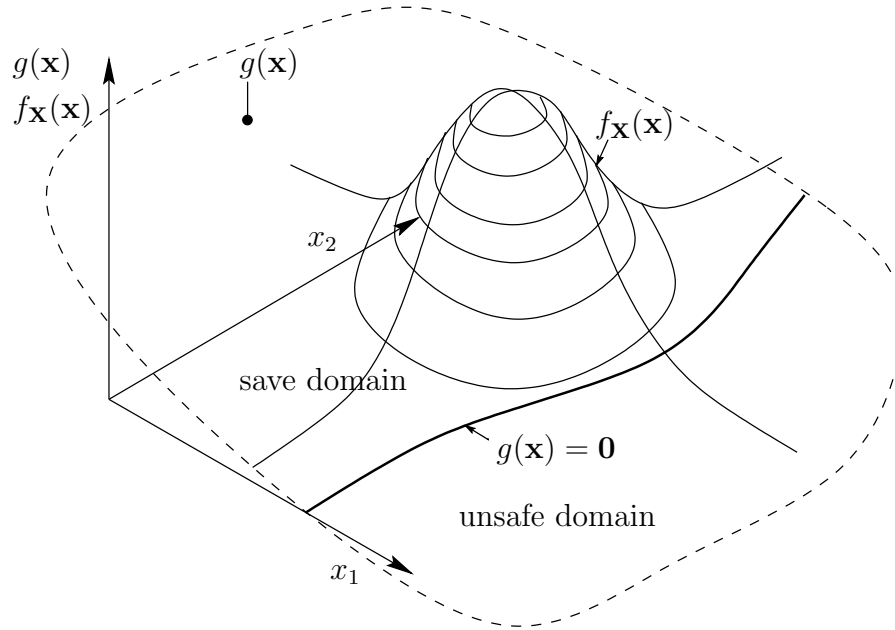


Figure 4: The state function  $g(\mathbf{x})$  of a numerical model is given implicitly, e.g. is result of a finite element analysis depending on several design responses. The failure condition leads to a unknown deterministic limit state function  $g(\mathbf{x}) = 0$ , where  $f_{\mathbf{X}}(\mathbf{x})$  is the joint probability density function.

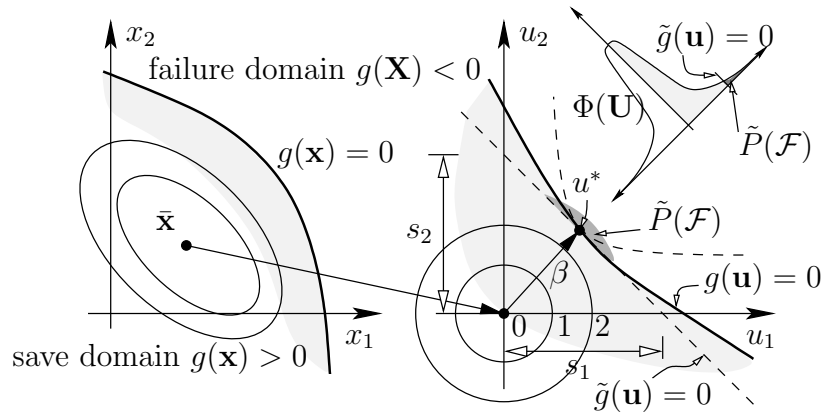


Figure 5: Transformations from correlated non-Gaussian variables  $\mathbf{X}$  to uncorrelated Gaussian variables  $\mathbf{U}$  with zero mean and unit variance. Gradient-based optimization for design point search (most probable point  $\beta$ -point). Approximation of the limit state function  $g(\mathbf{u})$  by a linear function  $\tilde{g}(\mathbf{u})$  in standard Gaussian space.

within the failure domain. Geometrically, this coincides with the point in the failure domain having the minimum distance  $\beta$  from the origin. From a safety engineering point of view, the point  $\mathbf{x}^*$  corresponding to  $\mathbf{u}^*$  is called *most probable failure point* or *design point*.

FORM (Shinozuka (1983); Hasofer & Lind (1974); Hohenbichler & Rackwitz (1988); Tvedt (1983); Breitung (1991)) is one of the most effective methods in reliability analysis, since the number of required function evaluations is relatively small.

In order to determine the failure probability  $P(\mathcal{F})$  in Eq. (2) the limit state function  $g(\mathbf{u})$  can be approximated by a Taylor expansion  $\bar{g}(\mathbf{u})$ . The optimal expansion point is the design point with coordinates  $(s_1, \dots, s_N)$ , which is the point on  $g(\mathbf{u})$  closest to the origin in Gaussian space. The distance  $\beta$  to the origin is called the reliability index.

Hence transformations from correlated non-Gaussian variables  $X$  to uncorrelated Gaussian variables  $U$  with zero mean and unit variance are required.

From the geometrical interpretation of the expansion point  $\mathbf{u}^*$  in standard Gaussian space it becomes quite clear that the calculation of the design point can be reduced to an optimization problem:

$$\mathbf{u}^* : \mathbf{u}^T \mathbf{u} \rightarrow \text{Min.}; \quad \text{subject to: } g[\mathbf{x}(\mathbf{u})] = 0 \quad (3)$$

This leads to the Lagrange-function

$$L = \mathbf{u}^T \mathbf{u} + \lambda g(\mathbf{u}) \rightarrow \text{Min.} \quad (4)$$

Standard optimization procedures can be utilized to solve for the location of  $\mathbf{u}^*$ . In **optiSLang** the NLPQLP algorithm is used, which is based on a sequential quadratic programming (SQP) method. Within this method the gradients of the objective function and the restrictions need to be determined, which is performed using finite differences. For further details see of the optimization method see Schittkowski (2004).

In the next step, the exact limit state function  $g(\mathbf{u})$  is replaced by a linear approximation  $\bar{g}(\mathbf{u})$  as shown in Fig. 2.2. From this, the probability of failure is easily determined to be

$$P(\mathcal{F}) = P[Z \leq 0] \approx P[\bar{Z} \leq 0] \quad (5)$$

$$= \int_{-\infty}^0 f_{\bar{Z}}(\bar{z}) d\bar{z} = F_{\bar{Z}}(0) \quad (6)$$

$$= \Phi\left(\frac{-E[\bar{Z}]}{\sigma_{\bar{Z}}}\right) = \Phi\left(-\frac{1}{\beta}\right) \quad (7)$$

$$= \Phi(-\beta) \quad (8)$$

This result is exact, if  $g(\mathbf{u})$  is actually linear. In general FORM gives a good approximation of the failure probability.

## 2.4 Plain Monte Carlo Simulation

The most general method to solve stochastic problems in structural mechanics is the well established Monte Carlo simulation method (for details see e.g. Rubinstein (1981)). However, the major shortcoming of this approach is its vast need of computational resources (the number of finite elements runs required) which cannot be provided in general situations.

Generally the failure probability is defined as the probability that the limit state function attains non-positive values:

$$P(\mathcal{F}) = P[g(X_1, X_2, \dots, X_n) \leq 0] \quad (9)$$

This can be written as an expected value:

$$P(\mathcal{F}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} I_g(x) f_{X_1 \dots X_n} dx_1 \dots dx_n \quad (10)$$

in which  $I_g(x_1 \dots x_n) = 1$  if  $g(x_1 \dots x_n) \leq 0$  and  $I_g(\cdot) = 0$  else.

$\beta$ level	Probability of failure $P(\mathcal{F})$	Number of required samples $N$ using PMC	Number of required samples $N$ using LHS
1	$1.59 \cdot 10^{-1}$	630	63
1.5	$6.69 \cdot 10^{-2}$	1,496	150
2	$2.28 \cdot 10^{-2}$	4,395	440
2.5	$6.21 \cdot 10^{-3}$	16,103	1,610
3	$1.35 \cdot 10^{-3}$	74,075	7,408
3.5	$2.33 \cdot 10^{-4}$	429,787	42,979
4	$3.17 \cdot 10^{-5}$	3,155,964	315,596
4.5	$3.40 \cdot 10^{-6}$	29,404,819	2,940,482
5	$2.87 \cdot 10^{-7}$	348,304,627	34,830,463

Table 2: Number of required samples to obtain a statistical error  $\sigma_{\bar{P}(\mathcal{F})}/E[\bar{P}(\mathcal{F})] = 10\%$  using plain Monte Carlo simulation (PMC) and latin hypercube sampling (LHS).

The analytical evaluation of this integral is in most practical problems impossible. This is due to the fact, that an exact determination of the primitive for many functions  $I_g(x)f_{X_1 \dots X_n}$  is not possible, or that no analytical expression for the limit state function  $g(\mathbf{x})$  can be derived, e.g. if  $g(\mathbf{x})$  is approximated by a limited number of complex FE-solutions.

In order to determine  $P(\mathcal{F})$  in principle all available statistical methods for estimation of expected values are applicable with

$$P(\mathcal{F}) = E[I_g(\mathbf{x})] \quad (11)$$

Samples are generated with respect to the probability density function of each variable and for each sample the response of the structure is determined. If  $N$  independent samples  $\mathbf{x}^{(k)}$  of the random vector  $\mathbf{X}$  are available then the estimator

$$\bar{P}(\mathcal{F}) = \frac{1}{N} \sum_{k=1}^N I_g(\mathbf{x}^{(k)}) \quad (12)$$

yields a consistent and unbiased estimate for  $P(\mathcal{F})$ . But in case of small values of  $P(\mathcal{F})$  and small values of  $N$  the confidence of the estimate is very low. The variance  $\sigma_{\bar{P}(\mathcal{F})}^2$  of the estimate  $\bar{P}(\mathcal{F})$  can be determined from

$$\begin{aligned} \sigma_{\bar{P}(\mathcal{F})}^2 &= E[(\bar{P}(\mathcal{F}) - E[\bar{P}(\mathcal{F})])^2] \\ &= \frac{P(\mathcal{F}) - (P(\mathcal{F}))^2}{N} \end{aligned}$$

and especially for small failure probabilities

$$\sigma_{\bar{P}(\mathcal{F})}^2 \approx \frac{P(\mathcal{F})}{N}$$

The required number  $N$  of simulations is independent of the dimension  $n$  of the problem.

Unfortunately for small failure probabilities  $P(\mathcal{F})$  a large number of samples is necessary in order to obtain a good estimator. Depending on the statistical error  $\epsilon = \sigma_{\bar{P}(\mathcal{F})}/E[\bar{P}(\mathcal{F})]$  the number of required samples is

$$N = \frac{1}{\epsilon^2 P(\mathcal{F})}$$

Assuming that the statistical error  $\epsilon = 0.1$  the required number is given by

$$N = \frac{100}{P(\mathcal{F})}$$

For small failure probabilities and complex problems with a computational expensive algorithm for the calculation of a single sample the evaluation of such a large number of samples is not acceptable (Table 2).

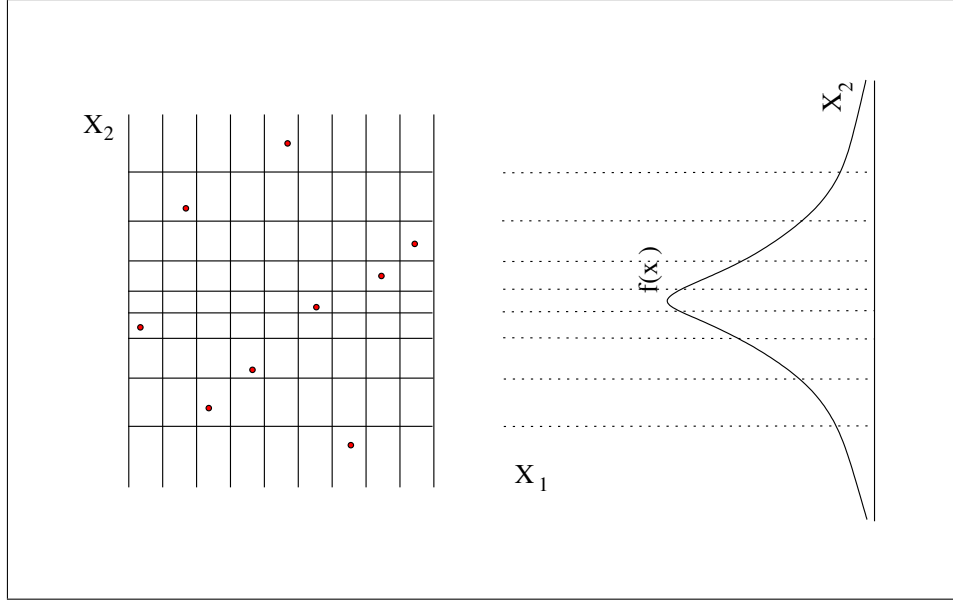


Figure 6: Latin hypercube sampling.

## 2.5 Latin Hypercube Sampling

Latin hypercube sampling was introduced by McKay *et al.* (1979). The domain of each random variable is decomposed into intervals with equal probability. The number of intervals corresponds to the number of samples. One value from each interval is selected at random with respect to the probability density in the interval. Combining the intervals of a random variables, the so called hypercubes are formed. This is illustrated in Figure 6 and in a similar way the representative point within one hypercube. The  $N$  representative values obtained for  $X_1$  are paired in a random manner (equally likely combinations) with the  $n$  values of  $X_2$  and so on until  $X_n$ . These  $N$  random combinations are called the latin hypercube samples. If the samples are combined in a random manner, artificial correlations are introduced, which can be avoid by regrouping the samples Iman & Conover (1982); Stein (1987). An estimate of the failure probability can be calculated by

$$\bar{P}(\mathcal{F}) = \frac{1}{N} \sum_{k=1}^N I_g(\mathbf{x}^{(k)})$$

The required number  $N$  of simulations is independent of the dimension  $n$  of the problem. Assuming that the statistical error  $\epsilon = 0.1$  number of required samples should be

$$N \geq \frac{10}{P(\mathcal{F})}$$

wherein  $P(\mathcal{F})$  is the expected probability of failure, as shown in Table 2).

## 2.6 Importance Sampling (Weighted Simulation)

### 2.6.1 General Concept

In order to reduce the standard deviation  $\sigma_{\bar{P}(\mathcal{F})}$  of the estimator to the order of magnitude of the probability of failure itself  $N$  must be in the range of  $N = 1/P(\mathcal{F})$ . For values of  $P(\mathcal{F})$  in the range of  $10^{-6}$  this cannot be achieved. Alternatively, strategies are employed which increase the "hit-rate" by artificially producing more samples in the failure domain than should occur according to the distribution functions, as shown in Figure 7. One way to approach this solution is the introduction of a positive weighting function  $h_{\mathbf{Y}}(\mathbf{x})$  which can be interpreted as density function of a random vector  $\mathbf{Y}$ . Samples

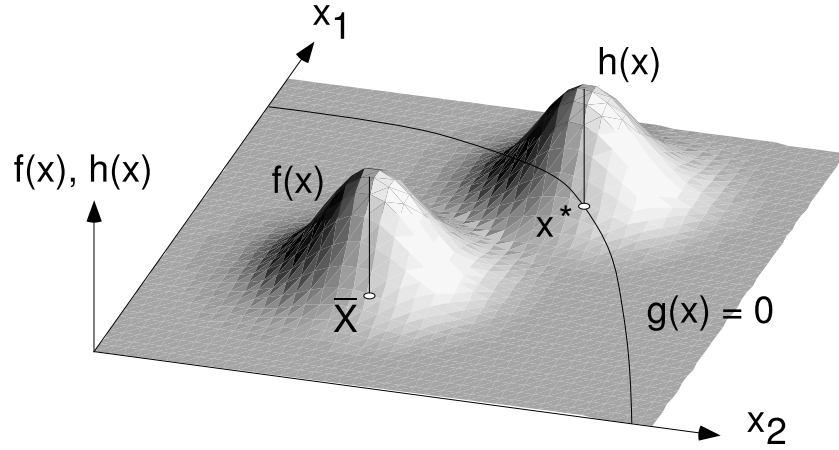


Figure 7: Original and importance sampling probability density functions.

are taken according to  $h_{\mathbf{Y}}(\mathbf{x})$ . The probability of failure is calculated with

$$P(\mathcal{F}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(g(\mathbf{x})) \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} h_{\mathbf{Y}}(\mathbf{x}) d\mathbf{x}$$

and the estimator is

$$\bar{P}(\mathcal{F}) = \frac{1}{N} \sum_{i=1}^N I(g(\mathbf{x}_i)) \frac{f_{\mathbf{X}}(\mathbf{x}_i)}{h_{\mathbf{Y}}(\mathbf{x}_i)} = E \left[ I(g(\mathbf{x})) \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} \right] \quad (13)$$

From the estimation procedure as outlined it is seen that the variance of the estimator  $\bar{P}(\mathcal{F})$  becomes

$$\sigma_{\bar{P}(\mathcal{F})}^2 = E[(\bar{P}(\mathcal{F}) - P(\mathcal{F}))^2] = E \left[ \left( \frac{1}{N} \sum_{i=1}^N I(g(\mathbf{x}_i)) \frac{f_{\mathbf{X}}(\mathbf{x}_i)}{h_{\mathbf{Y}}(\mathbf{x}_i)} - P(\mathcal{F}) \right)^2 \right]$$

A useful choice of  $h_{\mathbf{Y}}(\mathbf{x})$  can be based on minimizing  $\sigma_{\bar{P}(\mathcal{F})}^2$ . Ideally, such a weighting function should reduce the sampling error to zero which cannot be achieved in reality, since such a function should have the property (Rubinstein (1981)):

$$h_{\mathbf{Y}}(\mathbf{x}) = \begin{cases} \frac{1}{P(\mathcal{F})} f_{\mathbf{X}}(\mathbf{x}) & : g(\mathbf{x}) \leq 0 \\ 0 & : g(\mathbf{x}) > 0 \end{cases}$$

This property requires the knowledge of  $P(\mathcal{F})$  which, of course, is unknown. Special updating procedures such as adaptive sampling (Bucher (1988)) can help to alleviate this problem.

### 2.6.2 Importance Sampling Procedure Using Design Points (ISPUD)

A common possibility to determine the sampling density function within an importance sampling approach is used in the ISPUD algorithm of Bourgund & Bucher (1986). At first the design point is determined, which is analog to the FORM performed using NLPQLP. The mean value of the normal distributed sampling density function is the design point, whereas the variance is the same as the original density function. Although the method is applicable in Gaussian as well as in original space, it seems to be advantageous to perform the sampling in Gaussian space, especially if the random variables are correlated or not normal distributed. In this case the variance of the sampling density function (which equals 1 for the random variables transformed to Gaussian space) can be increased by a certain factor in order to spread the samples further from the design point. The failure probability is calculated using

Eq. (13). The advantage of this method is its independence with respect to the non linearity of the limit state function close to the design point, but similar to FORM the determination of the correct design point or the existence of multiple design points can cause errors. This concept can be accomplished in three steps:

- Determine the design point  $\mathbf{x}^*$  as shown in the context of the FORM-procedure.
- Choose a weighting function (sampling density)  $h_{\mathbf{Y}}(\mathbf{x})$  with the statistical moments  $E[\mathbf{Y}] = \mathbf{x}^*$  and  $\mathbf{C}_{\mathbf{Y}\mathbf{Y}} = \mathbf{C}_{\mathbf{X}\mathbf{X}}$  in the following form (multi-dimensional Gaussian distribution)

$$h_{\mathbf{Y}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det \mathbf{C}_{\mathbf{X}\mathbf{X}}}} \exp \left[ -\frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{C}_{\mathbf{X}\mathbf{X}}^{-1} (\mathbf{x} - \mathbf{x}^*) \right]$$

- Perform random sampling and statistical estimation according to Eq. (13).

The efficiency of this concept depends on the geometrical shape of the limit state function. In particular, limit state functions with high curvatures or almost circular shapes cannot be covered very well.

### 2.6.3 Adaptive Sampling

Another possibility to determine the sampling density function within an importance sampling approach is used in the adaptive sampling algorithm. This procedure, introduced by [Bucher \(1988\)](#), is based on the importance sampling technique, which concentrates the samples within the regions close to the failure domain.

Since the failure probability  $P(\mathcal{F})$  is a priori the unknown variable, the optimal sampling density function  $h_{\mathbf{Y}}(\mathbf{x})$  is approximated as follows. The sampling density function for the first iteration step is either the original density function  $f_{\mathbf{X}}(\mathbf{x})$  or an enlarged density function, where the variance of the original function is multiplied by a factor between 1 and 3 in order to have more samples within the failure domain. The latter approach is especially suitable for small failure probabilities.

After the first iteration the new sampling density function  $h_{\mathbf{Y}}(\mathbf{x})$  is determined, so that the first and second moments coincide with the ones of the samples, that are within the failure domain:

$$E[\mathbf{Y}] = E[\mathbf{X}|g(\mathbf{x}) \leq 0] \quad (14)$$

$$E[(\mathbf{Y} - \bar{\mathbf{Y}})(\mathbf{Y} - \bar{\mathbf{Y}})^T] = E[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^T | g(\mathbf{x}) \leq 0] \quad (15)$$

This iteration process is repeated several times. In practise 3 iterations are in general sufficient.

The problem of this method is a robust estimation of the covariance matrix in Eq. (15). If the number of samples within the failure domain is relatively small, which can be either due to a small number of total samples or a small failure probability, the covariance matrix is often not positive definite, and as a result the transformation of  $\mathbf{Y}$  to uncorrelated random variables is not possible.

In order to avoid this problem, a restriction for the number of samples  $N$  is introduced:

$$N_{\mathbf{x}|g(\mathbf{x}) \leq 0}^i > n^3 \quad (16)$$

where  $N_{\mathbf{x}|g(\mathbf{x}) \leq 0}^i$  is the number of samples in the failure domain for the  $i$ -th iteration and  $n$  the number of random variables. This restriction means, that at least  $n^3$  samples are required to reducing the statistical uncertainty within the calculation of the covariance matrix of the adaptive joint probability density function.

Usually, an initial variance multiplier greater than 3 leads to numerical errors calculating  $h_{\mathbf{Y}}(\mathbf{x})$ . Otherwise a small factor tends to an insufficient number of samples in the unsafe domain. A recommended choice of the initial variance multiplier and the initial samples size  $N^{(i=1)}$  could be the following procedure. In order to reduce the initial sample size the simulated initial probability of failure in case of random variables on the Gaussian space should be sufficient large, for example

$$P_{sim}^i(\mathcal{F}) = \Phi(-\beta) = 1 - \Phi\left(\frac{\beta - 0}{\sigma^{(i=1)}}\right) = 0.08 \dots 0.16$$

$P(\mathcal{F})$	$1.587 \cdot 10^{-1}$	$2.275 \cdot 10^{-2}$	$1.35 \cdot 10^{-3}$	$3.169 \cdot 10^{-5}$	$3.401 \cdot 10^{-6}$
$n/\beta$	1	2	3	4	4.5
2	1/63	1.8/63	2.5/70	3/90	3/120
5	1/790	1.8/920	2.5/1100	3/1400	3/1900
10	1/6250	1.8/7300	2.5/9000	3/11000	3/15000
15	1/21093	1.8/25000	2.5/30000	3/37100	3/50600
20	1/50000	1.8/60000	2.5/70000	3/90000	3/120000

Table 3: Initial variance multiplier  $\sigma^{(i=1)}$  and minimal initial samples size  $N^{(i=1)}$  depending on  $\beta$  and the number of random variables  $n$ .

So the initial variance multiplier in Gaussian space is equal to the initial standard deviation  $\sigma^{(i=1)}$  and should be

$$1 \leq \sigma^{(i=1)} = \frac{\beta}{\Phi^{-1}(1 - P_{sim}^i)} \leq 3$$

So the number of the initial number of samples  $N^{(i=1)}$  has to be greater than

$$N^{(i=1)} > \frac{N_{\mathbf{x}|g(\mathbf{x}) \leq 0}^{(i=1)}}{P_{sim}^i}$$

and minimal  $N^{(i=1)} > 63$  using latin hypercube sampling for each iteration step. Obviously an estimation of the  $\beta$  level or the failure probability itself is a priori required to determine the number of samples for the initial iteration, which might in many cases not be given. An approximation can either be obtained by engineering experience or by other methods as e.g. FORM. If in any iteration step this condition is violated, the adaptive procedure is stopped and the failure probability is calculated from the current iteration step.

Depending on the expected most probable failure point  $\beta$  and the number of random variables  $n$  the initial variance multiplier  $\sigma^{(i=1)}$  together with the minimal initial samples size  $N^{(i=1)}$  of the Table 3 are useful.

## 2.7 Response Surface Methods

### 2.7.1 Global Polynomial Approximation

Although Monte Carlo methods are most versatile, intuitively clear, and well understood, the computational cost (the number of finite element runs required) is in many cases prohibitive. Thus approximations become important which can be based e.g. on the response surface method. Normally, the state function  $g(\mathbf{X})$  of a system response is described implicitly, e.g. through an algorithmic procedure within finite element analysis.

Alternatively, the original state function can be approximated by a response surface function  $\tilde{g}(\mathbf{x})$  which has polynomial form (Rackwitz (1982); Faravelli (1986); Bucher & Bourgund (1987, 1990); Engelund & Rackwitz (1992); Rajashekhar & Ellingwood (1993)).

A commonly used method for response value approximation is the regression analysis. Usually, the approximation function is a first order or second order polynomial (Box & Draper (1987); Myers (1971)). As an example in the  $(n = 2)$ -dimensional case,  $k$ -responses ( $k = 1, \dots, m$ ) will be approximated using a least square quadratic polynomial in the following form:

$$\tilde{g}_k(\mathbf{x}) = \beta_1 x_{1k} + \beta_2 x_{2k} + \beta_{11} x_{1k}^2 + \beta_{22} x_{2k}^2 + 2\beta_{12} x_{1k} x_{2k} + \epsilon_k \quad (17)$$

Herein the term  $\epsilon_k$  represents the approximation errors. The approximate coefficients  $\beta$  can be calculated using the least square postulate

$$S = \sum_{k=1}^m \epsilon_k^2 = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} \rightarrow \min$$



Additional, the limit state function  $g(\mathbf{x}) = 0$  themselves can be interpolated by second order polynomials (Ouyornprasert & Bucher (1988); Bucher *et al.* (1988)).

One of the major advantages of the response surface method lies in its potential to selectively determine the number of structural analyses of the support points. This is especially helpful if some overall knowledge on the system behavior - particularly near to the failure region - is a priori available. By such means the computational effort can be substantially reduced.

On the other hand, the global approximation schemes widely used in the application of the response surface method can be quite misleading due to the lack of information in certain regions of the random variable space. Standard second order polynomial approximations are not sufficiently flexible. So, the estimation of the failure probability using this global approximation leads to large errors, in particular for small failure probabilities  $P(\mathcal{F}) < 10^{-2}$  and a number of random parameters of  $n > 5$ . It is therefore required to avoid such undesirable approximation errors at reasonable computational effort.

### 2.7.2 Adaptive Response Surface Method

**Moving least square approximation** A commonly used approximation method with minimized the regression error within the support point values is the moving least square method. The main advantage of this method is the flexibility for the approximation of highly nonlinear state and limit state functions. The proposed method is suitable for computing the reliability of complex models and is intended to provide reasonably accurate estimates of failure probabilities while maintaining computational efficiency.

Moving least square (MLS) functions can approximate locally clustered support point samples with higher local approximation quality. In addition MLS improve the response surface model using additional support points. MLS is formulated as

$$\hat{y}(\mathbf{x}) = \sum_{i=1}^{n_b} h_i(\mathbf{x}) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}) \mathbf{a}(\mathbf{x}) \quad (18)$$

with a predefined number of basis terms  $n_b$ , a vector of basis functions  $\mathbf{h}$  and the associated vector of the coefficients  $\mathbf{a}$ . Lancaster & Salkauskas (1986) formulates a local MLS approximation as

$$\hat{y}(\mathbf{x}, \mathbf{x}_j) = \sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}_j) \mathbf{a}(\mathbf{x})$$

with  $j = 1, \dots, n_s$  support points. The approximate coefficient vector  $\mathbf{a}$  can be calculated using the weighted least square postulate

$$\begin{aligned} S(\mathbf{x}) &= \sum_{j=1}^{n_s} w(\mathbf{x} - \mathbf{x}_j) (\hat{y}(\mathbf{x}, \mathbf{x}_j) - y(\mathbf{x}_j))^2 \\ &= \sum_{j=1}^{n_s} w(\mathbf{x} - \mathbf{x}_j) \left( \sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) - y(\mathbf{x}_j) \right)^2 \\ &= (\mathbf{H}\mathbf{a} - \mathbf{g})^T \mathbf{W}(\mathbf{x}) (\mathbf{H}\mathbf{a} - \mathbf{g}) \rightarrow \min \end{aligned} \quad (19)$$

with the weighting function  $w(\mathbf{x} - \mathbf{x}_j)$  and

$$\begin{aligned} \mathbf{g} &= [y(\mathbf{x}_1) \quad y(\mathbf{x}_2) \quad \dots \quad y(\mathbf{x}_{n_s})]^T \\ \mathbf{H} &= [\mathbf{h}(\mathbf{x}_1) \quad \mathbf{h}(\mathbf{x}_2) \quad \dots \quad \mathbf{h}(\mathbf{x}_{n_s})]^T \\ \mathbf{h}(\mathbf{x}_j) &= [h_1(\mathbf{x}_j) \quad h_2(\mathbf{x}_j) \quad \dots \quad h_{n_b}(\mathbf{x}_j)]^T \\ \mathbf{W}(\mathbf{x}) &= \text{diag}[w(\mathbf{x} - \mathbf{x}_1) \quad w(\mathbf{x} - \mathbf{x}_2) \quad \dots \quad w(\mathbf{x} - \mathbf{x}_{n_s})] \end{aligned}$$

The least square error  $S(\mathbf{x})$  may be a minimum in case that the partial gradients are zero.

$$\frac{\partial S(\mathbf{x})}{\partial \mathbf{a}} = 0$$

So using the equation (19) a linear equation system gives an estimation of the coefficient vector  $\mathbf{a}$

$$\mathbf{a}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{g} \quad (20)$$

with

$$\begin{aligned} \mathbf{M}(\mathbf{x}) &= \mathbf{H}^T \mathbf{W}(\mathbf{x}) \mathbf{H} \\ \mathbf{B}(\mathbf{x}) &= \mathbf{H}^T \mathbf{W}(\mathbf{x}) \end{aligned}$$

Cause the matrix of the basis function  $\mathbf{M}(\mathbf{x})$  should be non-singular always a sufficient number of  $n_s$  immediate neighbor support points have to be available. The number must be at least as large as number of the basis terms. The equation (20) inserted in (18) gives the approximation function

$$\hat{y}(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}) \mathbf{M}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{g}$$

An accurate as possible approximation quality requires a weighting function which is larger than zero  $w(\mathbf{x} - \mathbf{x}_j) > 0$  and monotonically decreasing  $w(\|\mathbf{x} - \mathbf{x}_j\|)$  inside of a small sub space  $\Omega_s \subset \Omega$ . So the influence of supports far from the actual coordinates is unimportant. A uniform weighting is given by a symmetry condition  $w(\mathbf{x} - \mathbf{x}_j) = w(\mathbf{x}_j - \mathbf{x}) = w(\|\mathbf{x} - \mathbf{x}_j\|)$ . Usually, an exponential function is used in this way:

$$w(\|\mathbf{x} - \mathbf{x}_j\|) = \begin{cases} e^{-\left(\frac{\|\mathbf{x} - \mathbf{x}_j\|}{D\alpha}\right)^2} & \|\mathbf{x} - \mathbf{x}_j\| \leq D \\ 0 & \|\mathbf{x} - \mathbf{x}_j\| > D \end{cases} \quad (21)$$

with a constant

$$\alpha = \frac{1}{\sqrt{-\log 0.001}}$$

and a influence radius  $D$  to choose. It is obvious that the smaller  $D$  the better the response values of the support points fit the given values. But as mentioned above at least  $n_b$  support points have to be available in every point to be approximated. Therefore it is possible that a  $D$  has to be chosen which leads to a large shape function error at the support points

To avoid these problems a new regularized weighting function was introduced by Most & Bucher (2005):

$$w_R(\|\mathbf{x} - \mathbf{x}_j\|) = \begin{cases} \frac{\hat{w}_R(\|\mathbf{x} - \mathbf{x}_j\|)}{\sum_{i=1}^{n_s} \hat{w}_R(\|\mathbf{x} - \mathbf{x}_i\|)} & \|\mathbf{x} - \mathbf{x}_j\| \leq D \\ 0 & \|\mathbf{x} - \mathbf{x}_j\| > D \end{cases} \quad (22)$$

with

$$\hat{w}_R(d) = \frac{\left(\left(\frac{d}{D}\right)^2 + \varepsilon\right)^{-2} - (1 + \varepsilon)^{-2}}{(\varepsilon)^{-2} - (1 + \varepsilon)^{-2}} ; \varepsilon \ll 1 \quad (23)$$

It is recommended by the authors to use the value

$$\varepsilon = 10^{-5}$$

This new regularized weighting function works better than the exponential function. But if the ratio of the minimal distance among the supports to the extent of areas where are no supports becomes worse the same problems occur again. As a matter of fact a large  $D$  is needed to approximate for coordinates where no support points are around and a small  $D$  is needed for coordinates where are a lot of support points in order to reach a minimal approximation error. To comply with this conditions it is necessary to use a function  $d(\mathbf{x})$  for the influence radius instead of a constant  $D$ .

Table 4: Numbers of required support points for different DOE schemes

Number of Variables $n$	Number of support points <sup>a</sup>						
	Linear approximation			Quadratic approximation <sup>b</sup>			
	Koshal Linear	D- optimal <sup>c</sup> (linear)	Full factorial ( $m = 2$ )	Koshal Quadr.	D- optimal <sup>d</sup> (quadr.)	Full factorial ( $m = 3$ )	Central composite (CCD)
1	2	2	2	3	3	3	3
2	3	4	4	6	9	9	9
3	4	6	8	10	15	27	15
4	5	8	16	15	23	81	25
5	6	9	32	21	32	243	43
6	7	11	64	28	42	729	77
7	8	12	128	36	54	2187	143
8	9	14	256	45	68	6561	273
9	10	15	512	55	83	19683	531
10	11	17	1024	66	99	59049	1045
11	12	18	2048	78	117	177147	2071
12	13	20	4096	91	137	531441	4121
13	14	21	8192	105	158	1594323	8219
14	15	23	16384	120	180	4782969	16413
15	16	24	32768	136	204	14348907	32799

<sup>a</sup>Including only one center point ( $n_c = 1$ ).<sup>b</sup>Also usable with linear approximation approach.<sup>c</sup>Based on full factorial DOE ( $m = 2$ ), with 1.5 times linear koshal designs.<sup>d</sup>Based on full factorial DOE ( $m = 3$ ), with 1.5 times quadratic koshal designs.

**Adaptive design of experiment** In particular, these response surfaces can be adaptively refined to consistently increase the accuracy of the estimated failure probability. This is especially suitable for the reliability analysis of complex nonlinear structures. An arbitrary number of check points even in high local concentration can be used without approximation problems. Using deterministic design of experiment, the necessary number of support points become very high with an increasing number of random variables, as shown in Table 4.

To decrease the number of support points in an optimized way, the so called D-optimality criterion is used. A discussion of this criterion is presented by [Box & Draper \(1971\)](#). The effectiveness of a design in satisfying the minimum variance (D-Optimal) criterion is expressed by the D-Efficiency of the design. In [Myers & Montgomery \(1995\)](#), more exact specifications of the D-Optimal criteria and further criteria called alphabetic optimality criteria are described. However, the first design of experiment in the first iteration should explore the random space including safe and unsafe domain as accurate as possible. A possible approach is given in [Klingmüller & Bourgund \(1992\)](#) with

$$x_i = \bar{x}_i \pm f \sigma_{x_i}$$

whereby

$$f = \Phi^{-1}(P(\mathcal{F})) = 3, \dots, 5$$

is a factor depending on the assumed failure probability. [Bucher & Bourgund \(1987\)](#) give an efficient possibility to adaption a design of experiment in the next iterations with

$$\mathbf{x}_M = \bar{\mathbf{x}} + (\mathbf{x}_D - \bar{\mathbf{x}}) \frac{g(\bar{\mathbf{x}})}{g(\bar{\mathbf{x}}) - g(\mathbf{x}_D)}$$

with

$$\mathbf{x}_D = E[\mathbf{X}|g(\mathbf{x}) \leq 0]$$

as shown in Figure 2.7.2. This is achieved by a combination of random search strategies (based on the adaptive sampling approach) as well as deterministic search refinement. In such a way for the most

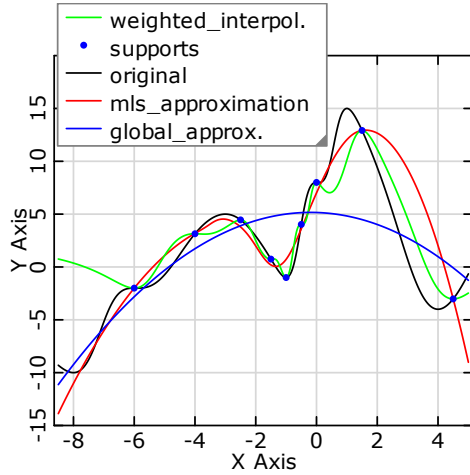


Figure 8: A comparison of different approximation functions: global 2nd order polynomial, moving least square with quadratic basis and constant basis (weighted interpolation).

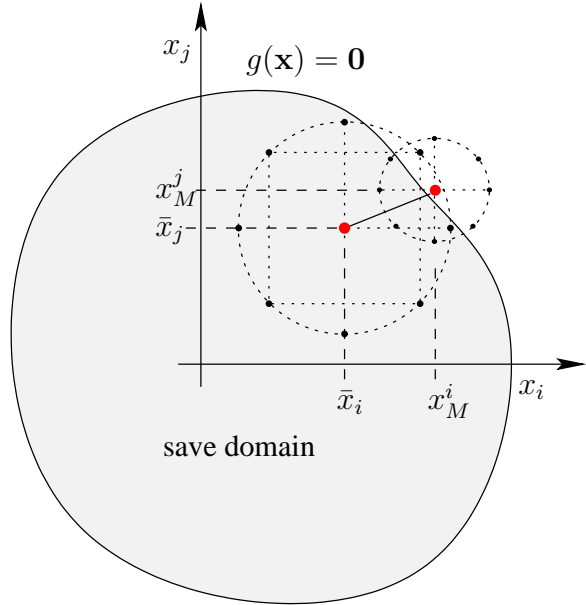


Figure 9: Adaptive design of experiment in the random space.

practical examples 3 till 6 iteration steps are necessary for a sufficient convergence. So this adaptive design of experiment using a D-optimal linear or quadratic design in combination with the improved moving least square approximation is suitable up to  $n \leq 20$  random parameters.

## 3 Examples

### 3.1 Introduction

In most practical applications the characteristics of the state function and the limit state function as e.g. its shape are not known a priori. Furthermore the limit state function is often not an analytic function but derived from a numerical model. It is therefore necessary to investigate the performance of the adaptive response surface method with respect to the following criteria:

- probability or  $\beta$  level
- number of random variables  $n$
- multiple  $\beta$ -points
- state function (high- or slow-curved, continuously differentiable, noisy)
- limit state function (high- or slow-curved, continuously differentiable, noisy)

### 3.2 The Quattro Function

Within the first example the different results of the presented adaptive response surface method in comparison with a classic global response surface method are given. The state function

$$g(x_1, x_2) = -\left(\frac{x_1}{6} + \frac{1}{2}\right)^4 - \left(\frac{x_2}{8} - \frac{6}{5}\right)^4 - \left(\frac{2}{9}x_1 + \frac{1}{2}\right)^3 \left(\frac{x_2}{8} - \frac{6}{5}\right) + \left(\frac{x_1}{6} + \frac{1}{2}\right)^2 + \left(\frac{x_2}{8} - \frac{6}{5}\right)^2 - \frac{5}{3}x_1 - \frac{3}{4}x_2 + \frac{11}{5}$$

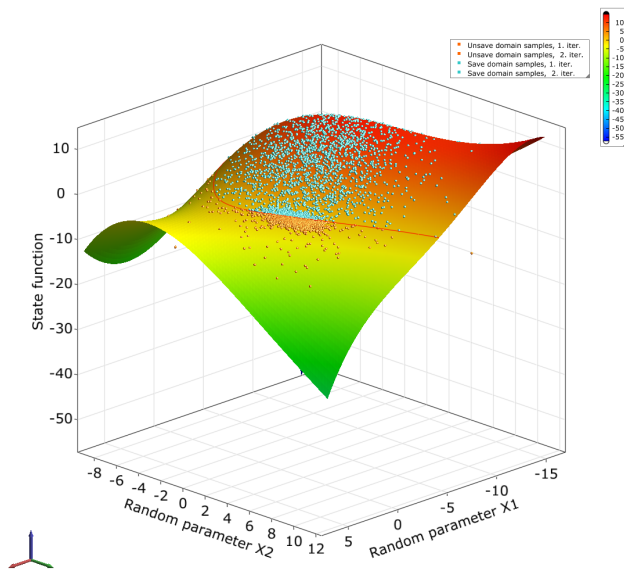


Figure 10: The Quattro Function. Analytical state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$  in combination with direct importance adaptive sampling.

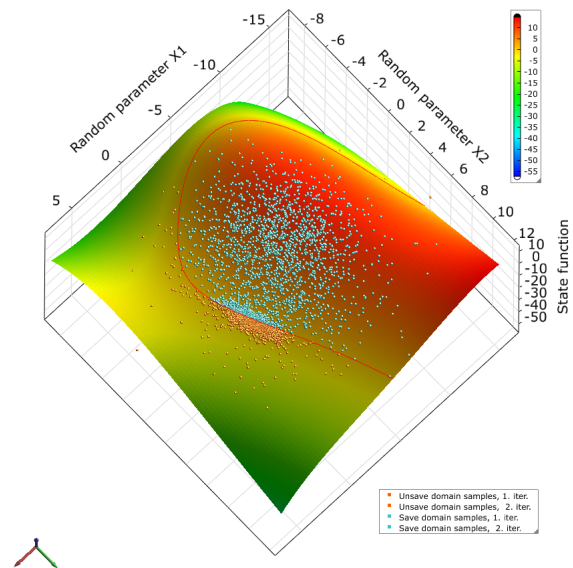


Figure 11: The limit state function  $g(\mathbf{x}) = 0$  is a high-curved nonlinear one. The random parameters  $X_1$  and  $X_2$  are normal distributed variables with mean  $\bar{X}_1 = -3.9$ , and  $\bar{X}_2 = 1.5$  and standard deviation  $\sigma_{X_1} = \sigma_{X_2} = 1$ .

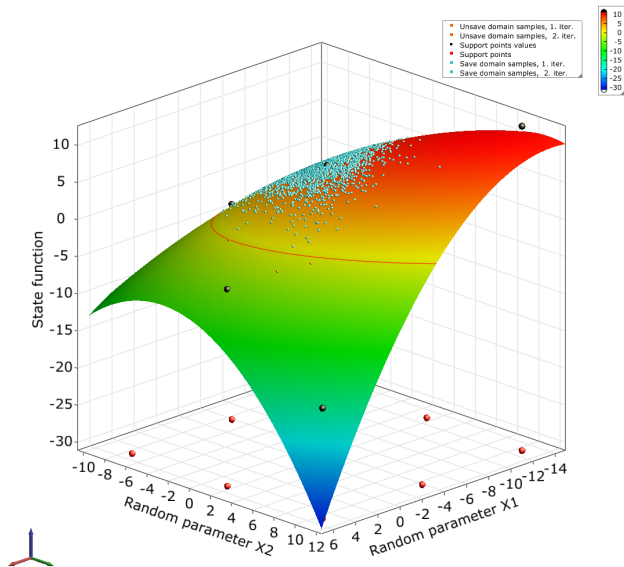


Figure 12: Approximated state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$  using the global 2nd order polynomial approximation using a global central composite design of experiment.

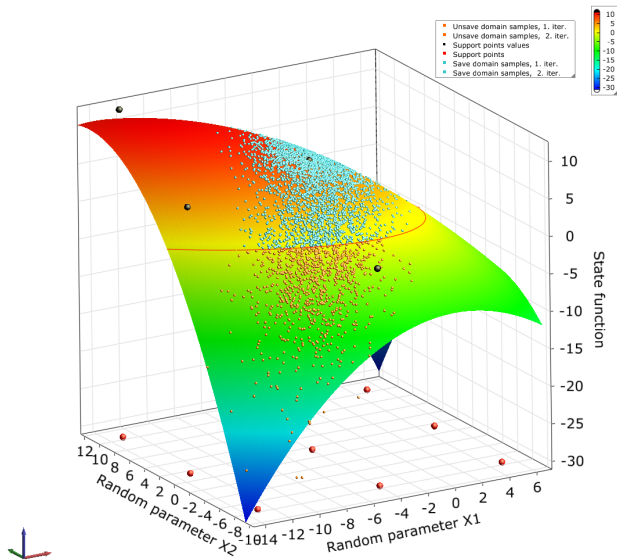


Figure 13: Using the global 2nd order polynomial approximation a wrong most probability failure domain is identified using importance adaptive sampling.

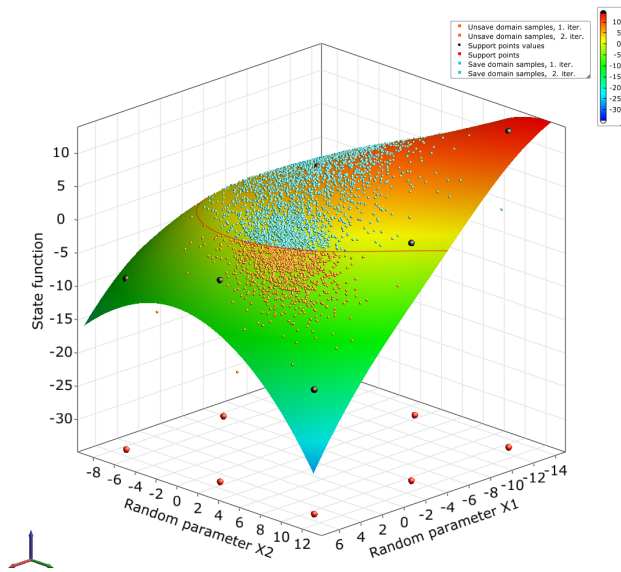


Figure 14: Approximated state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$  using the adaptive response surfaces with importance adaptive sampling and the first design of experiment (central composite design).

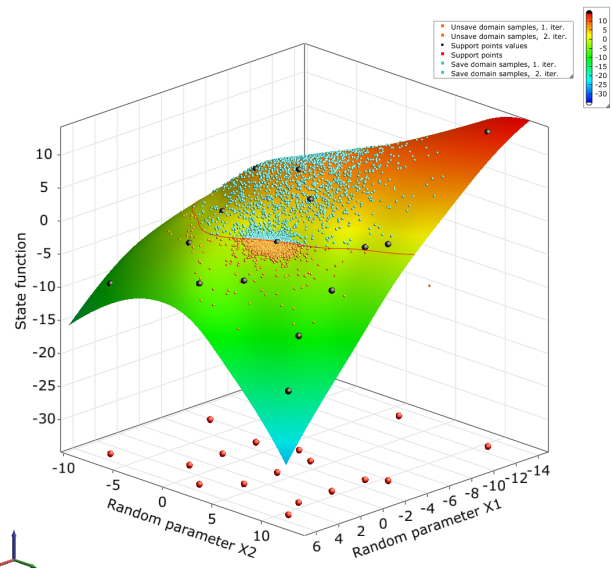


Figure 15: Adaptive moving least square approximation with first and second design of experiment.

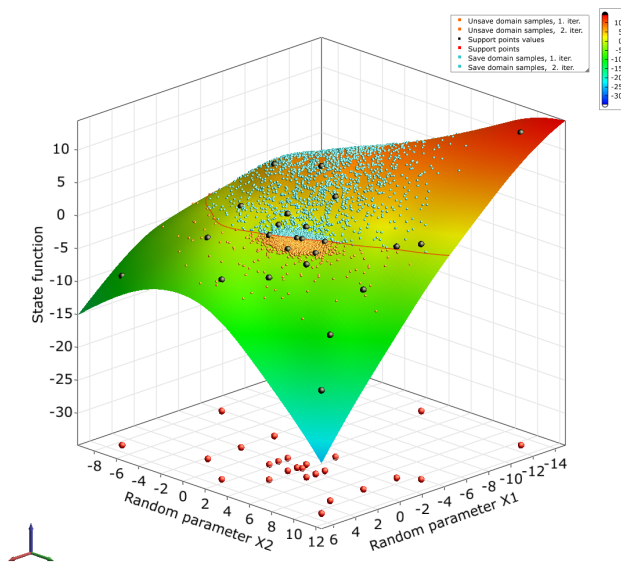


Figure 16: Adaptive moving least square approximation with first till third design of experiment.

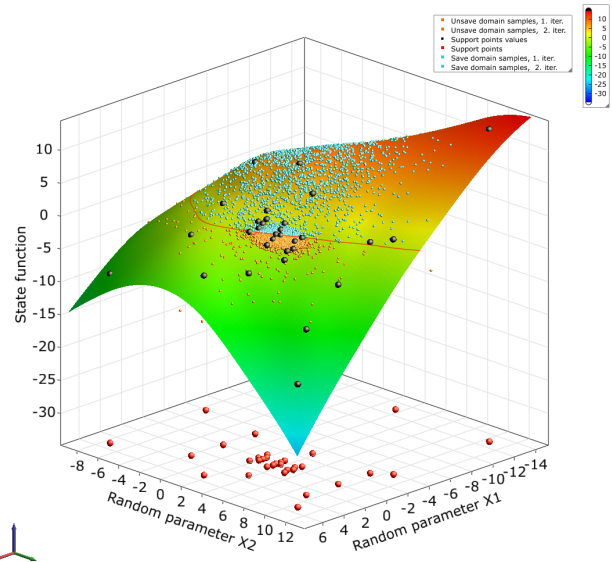


Figure 17: Adaptive moving least square approximation with first till fourth design of experiment.



	Number of state function evaluations	Failure probability	Variance of the estimation $\sigma_{\bar{P}(\mathcal{F})}^2$	Accuracy error %
Direct adaptive importance sampling (AS)	4000	$4.8907 \cdot 10^{-6}$	$1.5813 \cdot 10^{-7}$	0
Global polynomial 2nd order approximation (RSM)	9	$2.1563 \cdot 10^{-10}$	$2.5303 \cdot 10^{-11}$	2268000
Adaptive response surface approximation (ARSM)	9	$7.9728 \cdot 10^{-11}$	$6.0763 \cdot 10^{-12}$	6134100
	18	$4.47 \cdot 10^{-6}$	$1.1713 \cdot 10^{-7}$	9
	27	$4.4096 \cdot 10^{-6}$	$1.214 \cdot 10^{-7}$	11
	36	$4.6044 \cdot 10^{-6}$	$1.1337 \cdot 10^{-7}$	6

Table 5: Results of the failure probability of the Quattro Function depending on the applied analysis method.

	Number of state function evaluations	Failure probability	Variance of the estimation $\sigma_{\bar{P}(\mathcal{F})}^2$	Accuracy error %
Direct adaptive importance sampling (AS)	4000	$3.6817 \cdot 10^{-6}$	$3.5149 \cdot 10^{-7}$	0
Adaptive response surface approximation (ARSM)	25	$4.5943 \cdot 10^{-4}$	$2.6486 \cdot 10^{-5}$	99
	34	$1.05 \cdot 10^{-4}$	$4.2485 \cdot 10^{-6}$	96
	43	$3.3566 \cdot 10^{-6}$	$7.2673 \cdot 10^{-7}$	10
	52	$7.9314 \cdot 10^{-6}$	$1.0342 \cdot 10^{-6}$	54
	61	$3.7946 \cdot 10^{-6}$	$3.3768 \cdot 10^{-7}$	3

Table 6: Results of the failure probability of the Himmelblau Function depending on the applied analysis method.

is a two-dimensional fourth order polynomial, as shown in Figure 10. Furthermore, the limit state function  $g(\mathbf{x}) = 0$  is a high-curved nonlinear one, as shown in Figure 11. The random parameters  $X_1$  and  $X_2$  are normal distributed variables with mean  $\bar{X}_1 = -3.9$ , and  $\bar{X}_2 = 1.5$  and standard deviation  $\sigma_{X_1} = \sigma_{X_2} = 1$ . In order to obtain the reference failure probability a direct importance adaptive sampling is done with  $N = 2000$  samples, two iterations and an initial variance multiplier  $\sigma^{(i=1)} = 3$ . The given failure probability is  $P(\mathcal{F}) = 4.8907 \cdot 10^{-6}$  with a standard error of  $\sigma_{\bar{P}(\mathcal{F})} = 1.5813 \cdot 10^{-7}$ . In addition, the same adaptive sampling procedure is used to calculate the failure probability on the response surfaces.

For this example, the approximation using the global 2nd order polynomial approximation and a global central composite design of experiment leads to an error of 2268000 % in calculation the failure probability, as shown in the Figures 12 and 13. Using the global 2nd order polynomial approximation a wrong most probability failure domain is identified using importance adaptive sampling.

Applying the new adaptive response surface method to this state function, as shown in Figures 14 till 17, leads to accurate estimation of the failure probability already after the first adaption with  $N = 18$  state function evaluations. In summary, using three adaptations of the central composite design with in total  $N = 36$  state function evaluations the error of the given failure probability is 6% only (see Table 5 for details).

### 3.3 The Himmelblau Function

A commonly used fourth order polynomial test function within the nonlinear optimization is the so-called Himmelblau (1972) function

$$g(x_1, x_2) = \left( \frac{x_1^2}{1.81} + \frac{x_2}{1.81} - 11 \right)^2 + \left( \frac{x_1}{1.81} + \frac{x_2^2}{1.81} - 7 \right)^2 - 50$$

with multiple separated unsafe domains. The Figure 18 shows the Himmelblau state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$ . The random parameters  $X_1$  and  $X_2$  are normal distributed variables

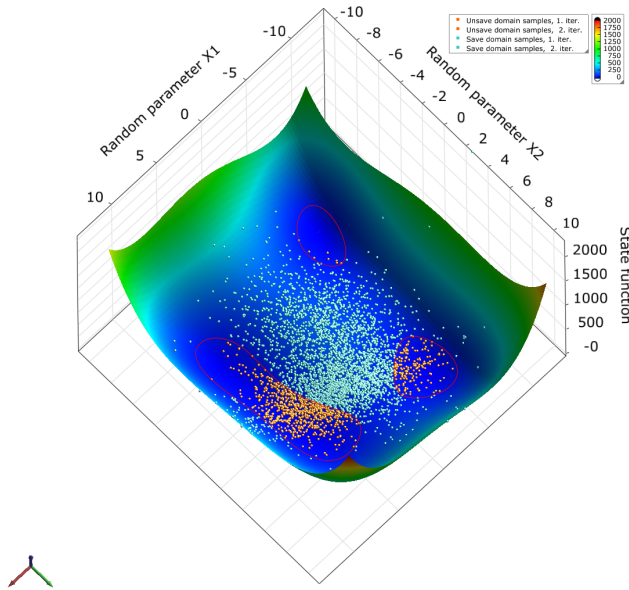


Figure 18: The Himmelblau Function. Analytical state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$  with direct importance adaptive sampling. The random parameters  $X_1$  and  $X_2$  are normal distributed variables with mean  $\bar{X}_1 = -0.6$ , and  $\bar{X}_2 = 0$  and standard deviation  $\sigma_{X_1} = \sigma_{X_2} = 1$ .

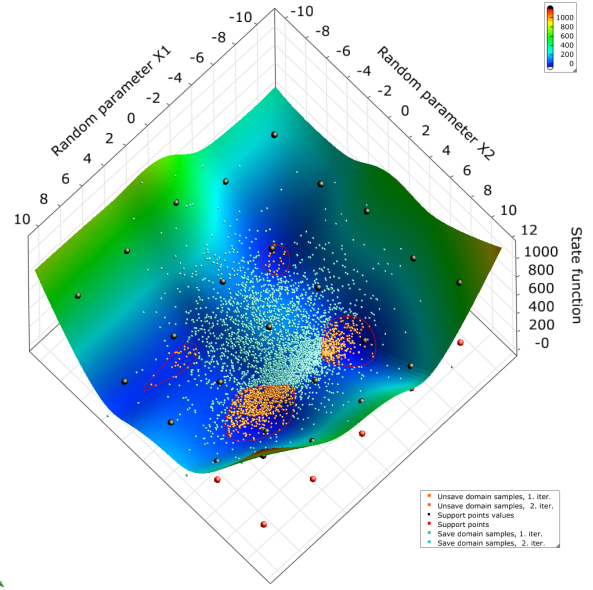


Figure 19: Approximated state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$  using the adaptive response surfaces with importance adaptive sampling and the first design of experiment (full factorial design).

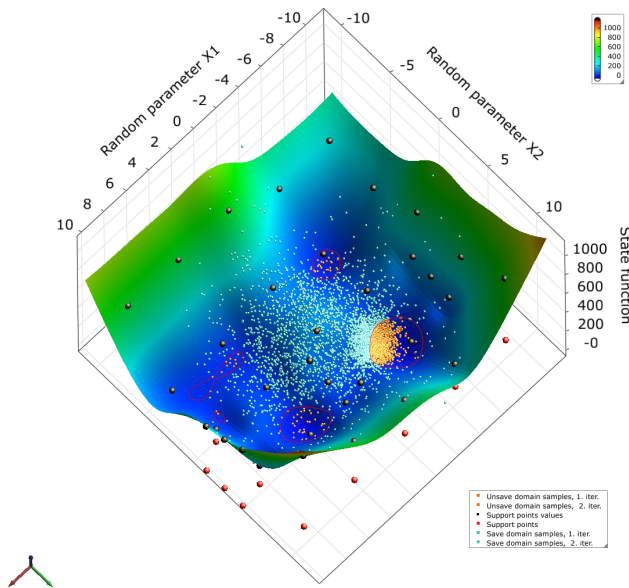


Figure 20: Adaptive moving least square approximation with first and second (central composite) design of experiment.

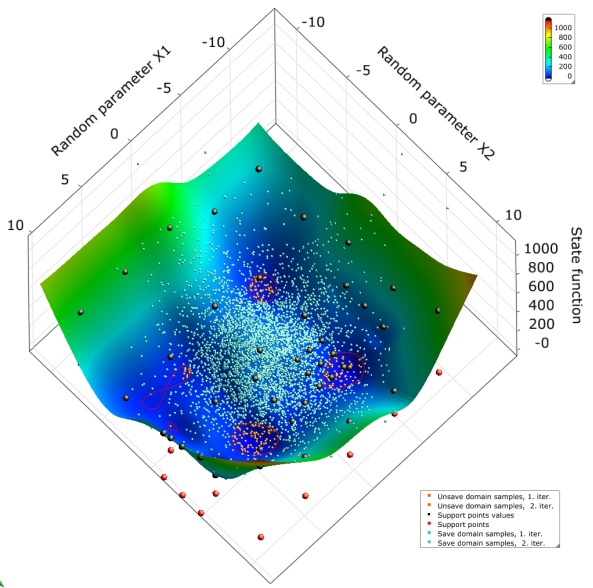


Figure 21: Adaptive moving least square approximation with first till third (central composite) design of experiment.



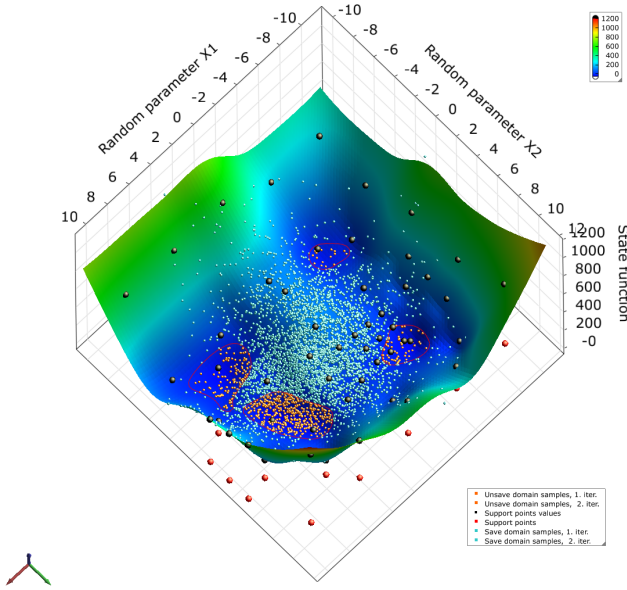


Figure 22: Adaptive moving least square approximation with first till fourth (central composite) design of experiment.

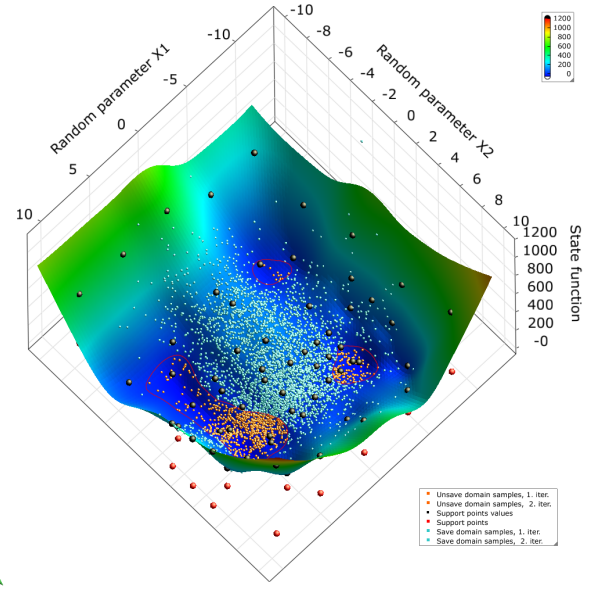


Figure 23: Adaptive moving least square approximation with first till fifth (central composite) design of experiment.

with mean  $\bar{X}_1 = -0.6$ , and  $\bar{X}_2 = 0$  and standard deviation  $\sigma_{X_1} = \sigma_{X_2} = 1$ . Using direct importance adaptive sampling the reference failure probability is  $P(\mathcal{F}) = 3.6817 \cdot 10^6$  with a standard error of  $\sigma_{P(\mathcal{F})} = 3.5149 \cdot 10^{-7}$ . Contingent on the existence of multiple separated unsafe domains other analysis methods existing in **optiSLang** like directional sampling (Bjerager (1988); Ditlevsen & Bjerager (1989); Melchers (1990)) or FORM and ISPUD are not applicable.

Applying adaptive response surface method to this state function, as shown in Figures 19 till 23, leads to accurate estimation of the failure probability already after the fourth adaption with in total  $N = 61$  state function evaluations. The Figure 19 shows the approximated state function  $g(\mathbf{x})$  and limit state function  $g(\mathbf{x}) = 0$  using the first design of experiment (full factorial design with  $N = 25$  state function evaluations). In summary, using four adaptions of the central composite design with  $N = 9$  state function evaluations per iteration the error of the given failure probability is 3% only (see Table 6 for details).

### 3.4 Nonparametric Structural Reliability Analysis

#### 3.4.1 Introduction

A cylindrical shell structure with imperfect geometry is studied. The imperfections are modelled as random field of the coordinates of the underlying finite element model. A robustness study is performed before the actual reliability analysis with the purpose of reducing the number of random variables. For reliability analysis, Monte Carlo simulation using Latin Hypercube sampling is utilized. The failure criterion is derived from a comparison of the linear buckling loads of the perfect and the imperfect structures. The computational finite element solver tasks are performed using the software package **SLang**, (Bucher *et al.* (1995)).

#### 3.4.2 Random Fields

**Properties** A random field is, in brief, a random function  $H(\mathbf{x})$  defined on a spatial structure. The vector  $\mathbf{x}$  points to a location on the structure. Random fields are used, e.g., to study random fluctuations in geometrical or material properties of a mechanical or structural system. In other words, the considered property is a random variable at each point on the structure. Moreover, the random properties at two different locations can be mutually correlated among each others.

Any random variable is described by a probability distribution function, which can be parameterized by distribution type and stochastic moments. For description of a random field, these properties become functions over space as well. *From now on, a Gaussian (or Normal) distribution type is assumed.* In this case, the description by first and second moments provides the full information. In particular

$$\mu_H(\mathbf{x}) = E[H(\mathbf{x})] = \int_{-\infty}^{+\infty} h f_H(\mathbf{x}, h) dh \quad (24)$$

denotes the *mean function*, and

$$R_{HH}(\mathbf{x}_1, \mathbf{x}_2) = E[H(\mathbf{x}_1) \cdot H(\mathbf{x}_2)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_1 h_2 f_H(\mathbf{x}_1, \mathbf{x}_2, h_1, h_2) dh_1 dh_2 \quad (25)$$

the *correlation function*, with  $E[.]$  being the expected value operation (Soong & Grigoriu (1993)).  $\mathbb{R}_x$  is the domain of admissible values of  $\mathbf{x}$ .  $R_{HH}$  is a function of the distance between two points and indicates the amount of linear dependency between the random properties at these locations. It has the properties of symmetry:

$$R_{HH}(\mathbf{x}_1, \mathbf{x}_2) = R_{HH}(\mathbf{x}_2, \mathbf{x}_1) \quad (26)$$

and positive semi-definiteness:

$$\iint_{\mathbb{R}_x \mathbb{R}_x} R_{HH}(\mathbf{x}_1, \mathbf{x}_2) w(\mathbf{x}_1) w(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \geq 0 \quad (27)$$

for any real valued function  $w(.)$  defined on the structure.

The so-called *correlation length*  $L_{HH}$ , which is actually the centre of gravity of the correlation function, is a typical characteristic of  $R_{HH}$ . It has to be estimated from manufacturing processes, natural scatter of material properties, etc. An infinite correlation length yields a structure with random properties, yet without fluctuations within the structure. A zero correlation length yields uncorrelated (in case of the Gaussian distribution independent) variables. It can be shown that the structural analysis yields the same results as for  $L_{HH} = \infty$ .

Two special cases are important for the further studies. *Homogeneity*: A random field is said to be homogeneous in the wide sense, if the first and second moments are the same at any possible location, i.e.

$$\mu_H(\mathbf{x}) = \mu_H(\mathbf{x} + \boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \quad (28)$$

$$R_{HH}(\mathbf{x}_1, \mathbf{x}_2) = R_{HH}(\mathbf{x}_1 + \boldsymbol{\xi}, \mathbf{x}_2 + \boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \quad (29)$$

*Isotropy* (in the wide sense) claims that the correlation function depends on the distance between the two observed locations  $\mathbf{x}_1, \mathbf{x}_2$  only, not on the direction:

$$R_{HH}(\mathbf{x}, \mathbf{x} + \boldsymbol{\xi}) = R_{HH}(\|\boldsymbol{\xi}\|) \quad (30)$$

**Modelling** For computational analyses, a random field has to be discretized in order to yield a finite set of random variables  $\mathbf{X}$ , which are assigned to discrete locations on the observed structure. Since the Finite Element Method is the standard for structural analyses, it is convenient to discretize the random field in the same way as the finite element model. One speaks of Stochastic Finite Elements in this case. The discretization can be oriented at the element mid points, integration points, or nodes. The properties of the random variables are derived from the random field properties explained previously. The spatial discretization should be able to model the variability of the random field. For this purpose, the distance between two nodes should be not more than  $1/4$  of  $L_{HH}$  (Der Kiureghian & Ke (1988); Hisada & Nakagiri (1981)).

The set of random variables is then characterized by a mean vector and a correlation matrix. It is convenient for the developments that follow to use the covariance matrix instead, which is defined as

$$\mathbf{C}_{XX} : \quad c_{ij} = R_{HH}(\mathbf{x}_i, \mathbf{x}_j) - \mu_H(\mathbf{x}_i) \cdot \mu_H(\mathbf{x}_j) \quad (31)$$

The joint density of all random variables can be modelled with help of the Nataf model (Nataf (1962); Liu & Der Kiureghian (1986)), given the type and properties of the marginal distributions for each variable.

From now on, *random fields with zero mean vector are considered*. Then the covariance matrix suffices for the characterization of the random variables set. Random number generators can produce independent random variables only. For the present case of a Gaussian distribution, independence is equivalent to uncorrelatedness. It can be shown that the random variables will be uncorrelated after the following transformation. The covariance matrix is decomposed with help of an eigenvalue analysis:

$$\mathbf{\Psi}^T \mathbf{C}_{XX} \mathbf{\Psi} = \text{diag}\{\lambda_i\} \quad (32)$$

Therein,  $\mathbf{\Psi}$  is the matrix of eigenvectors of  $\mathbf{C}_{XX}$  stored columnwise, and the eigenvalues are identical to the variances of the uncorrelated random variables  $Y_i$ :  $\lambda_i = \sigma_{Y_i}^2$ . The transformation rule reads

$$\mathbf{Y} = \mathbf{\Psi}^T \mathbf{X} \quad (33)$$

and the backward transformation

$$\mathbf{X} = \mathbf{\Psi} \mathbf{Y} \quad (34)$$

because the eigenvectors  $\mathbf{\Psi}$  form an orthonormal basis. Hence it is possible to simulate the random field with a set of uncorrelated random variables  $\mathbf{Y}$  and the respective (deterministic) shape functions  $\mathbf{\Psi}$ .

The eigenvalues are usually stored sorted by magnitude in descending order, which is a measure for their relevance in representing  $\mathbf{C}_{XX}$ . This opens a way of reducing the usually huge number of variables. Only the random variables with the highest variances are needed for simulation. The quality of approximation of the random field is expressed by the variability fraction (Brenner (1995))

$$Q = \frac{\sum_{i=1}^n \sigma_{Y_i}^2}{\text{trace}(\mathbf{C}_{XX})} ; \quad 0 \leq Q \leq 1 \quad (35)$$

The number of considered variables has to be adjusted before the simulation for a sufficient quality, e.g.  $Q > 0.9$ .

**Application: Cylindrical Shell** The concept of non-parametric reliability analysis shall be demonstrated by an application example. The structure considered is a cylindrical shell with properties as given in table 7. The cylinder has a Navier-type support at the bottom edge and is loaded along the top edge with a continuous vertical unit load. The structure is modelled with a 9-node isoparametric shell finite element type (SHELL9N) within the program `SPiG` (Bucher *et al.* (1995)). Figure 24 shows the finite element model with loads and restraints. The load, however, is applied as an equivalent prestrain of the elements, because of the difficult modelling and computing of in-plane loads for this element type.

A random field is applied on the structure in order to model geometrical imperfections. The random properties are coordinate deviations from the perfect structure in the cylinder's radial direction. Thus the random field is discretized at the nodes of the finite element mesh. It has zero mean and a standard deviation of  $\sigma_H = 10^{-3}$  mm, which is roughly a hundredth of the radius. The orthogonal field has different correlation functions along the perimeter and height as plotted in fig. 25. The spatial correlation structure with respect to one node on the bottom edge is visualized, too. While fig. 31 shows a few eigenvectors of the covariance matrix, a sampled imperfection shape can be seen in fig. 26, both are scaled.

Table 7: Properties of cylindrical shell structure.

Wall thckn. [mm]	Radius [mm]	Height [mm]	Young's mod. [N/mm <sup>2</sup> ]
0.197	101.6	139.7	$6.895 \cdot 10^4$

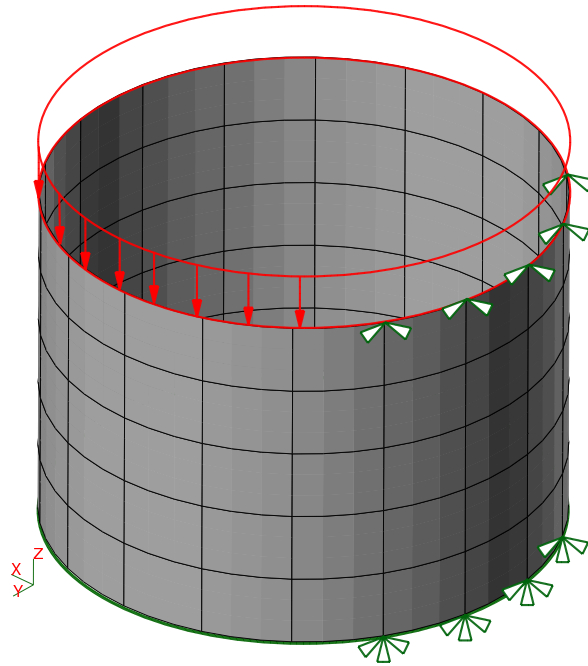


Figure 24: Finite element model of the cylindrical shell with schematic loads and supports.

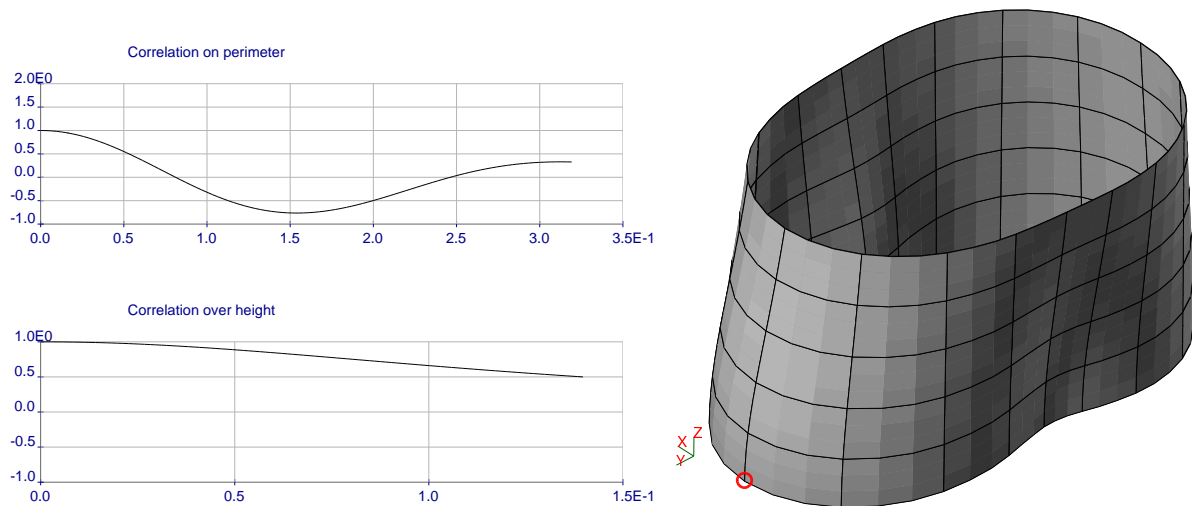


Figure 25: Correlation functions over perimeter (above) and height (below) and spatial correlation structure with respect to the marked node.

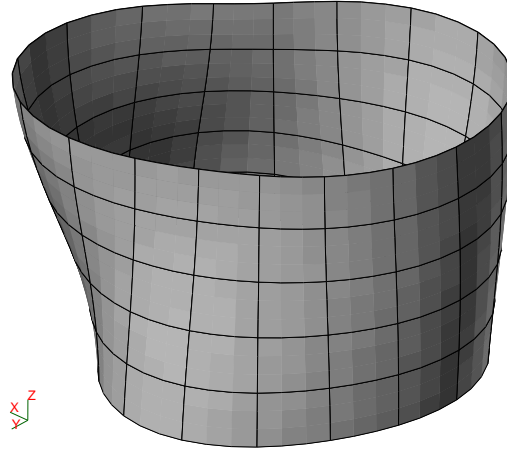


Figure 26: Sample of an imperfection shape (magnified).

The reliability towards stability failure is studied. The limit load is adopted from the buckling load of the perfect structure subtracted a “safety margin”. The limit state function reads

$$g(\mathbf{X}) = F_{buckling} - 34 \text{ kN} \leq 0 \quad (36)$$

**Robustness Analysis** The number of random variables used to model a random field can be very high, such that numerical difficulties may occur, particularly when the probability of failure is low. Experience with other applications of reliability analysis with geometrical imperfection (Schorling (1998)) shows, that the reduction of variables explained in section 3.4.2, eq. (35) fails, because the criterion is purely stochastic and does not account for the structural behaviour. On the other hand, purely mechanical considerations may not work as well, if they fail to represent the random field properly.

The program **optiSLang** (dynardo GmbH (2006)) offers the feature of robustness analysis. This is used to find a suitable selection of variables. In brief, the robustness analysis examines statistical dependencies between any input and output quantities the user desires. The data is obtained from a Monte Carlo simulation with small sample size. The input variables are simulated following statistical properties put in by the user, or they are varied systematically within given bounds. optiSLang offers functions such as filters, fit tests or a principal component analysis of the correlation matrix, which shall reveal the most relevant influences on the observed output quantities. Results are plotted as coloured matrices, histograms etc. which make it easy to identify dependencies between variables. Two functions are explained in more detail in the following.

By computing the *quadratic correlation* it is tested, if one variable  $Y$  can be represented by a quadratic regression of another variable,  $X$ . The regression model is

$$\hat{Y}(X) = a + bX + cX^2 \quad (37)$$

Samples of  $\hat{Y}$  are gained by inserting samples of  $X$  into eq. (37), values of  $Y$  itself are computed directly. Then the correlation coefficients  $\rho_{Y\hat{Y}}$  and  $\rho_{\hat{Y}Y} \neq \rho_{Y\hat{Y}}$  are evaluated. The values range from 0 to 1, high values indicate a strong quadratic correlation between  $X$  and  $Y$ .

The *coefficient of determination* is the ratio of the variance of a regression model to the original variable. It indicates the amount of variability of an output variable  $Y$ , which can be explained by the variability of the input variable  $X$  underlying the regression model. For the quadratic regression of eq. (37):

$$R^2 = \frac{\sum_{i=1}^n (\hat{Y}_i(X_i) - \mu_Y)^2}{\sum_{i=1}^n (Y_i - \mu_Y)^2} \quad (38)$$

Values of  $R^2$  vary from 0 to 1. A high value, e.g.  $R^2 = 0.8$ , means that 80 % of the variability of  $Y$  can be explained by a quadratic relation between  $Y$  and  $X$ . However, this is no accuracy measure of the regression model. While the (quadratic) correlation coefficients only give information about the mutual relation of two variables, the coefficients of determination allow for a comparison of the influences of all input variables to the output.

**Results** A robustness analysis is performed with **optiSLang**, where all 396 variables of the random field are considered and their relations to the critical buckling loads of the simulated imperfect structures are examined. Other output quantities were computed as well, such as characteristics of the simulated imperfection shape or the buckling vector, stresses etc. The purpose of these examinations was to find criteria for reducing the number of random variables or savings of FE calculations for irrelevant imperfections. Only the observations between the random variables as input and the critical buckling load as output are discussed in the following.

No significant linear correlations could be found. Instead, strong quadratic correlations are observed between the first 14 random variables (where “first” indicates those variables with the highest variances, i.e. highest eigenvalues after decomposition of the covariance matrix, cf. sect. 3.4.2) and the critical load. For variables of order higher than 14, the quadratic correlation coefficients are close to zero. The quadratic correlation matrix as explained in section 3.4.2 is displayed in fig. 27. The nonlinear dependency becomes obvious by an anthill plot, that is a plot of realizations of input and output variable pairs, see fig. 28 for an example. Fig. 29 shows a histogram of the sampled critical load and a probability density of Weibull type, which was the type of highest acceptance in the fit test. Since all input variables are of Gaussian type but the output is not, this is another evidence for the non-linear relation.

Based on the quadratic regression of eq. (37) for the buckling load, with each random variable set in for  $X$  successively, the coefficients of determination, eq. (38), are computed. The sum of all values is less than 100 %. That means, the variance of the critical load cannot be fully explained by a quadratic relation to the input variables. The results are sorted and plotted as bar diagram, fig. 30. The strongest influences can easily be identified. A closer look reveals that not all of the “first 14” variables (see above) are most relevant, but a few variables assigned to higher order eigenvectors of the covariance matrix as well. The eigenvectors of the covariance matrix (used as shape functions for the simulated random amplitudes in the reliability analysis) which are selected by the criteria of either the highest variances (“first 14”) or the highest coefficients of determination (“top 14”) are gathered in fig. 31. The selection by the latter criterion are the eigenvalues/eigenvectors of order 1, 2, 5, 6, 15, 21, 22, 26, 29, 30, 32, 34, 83, 197.

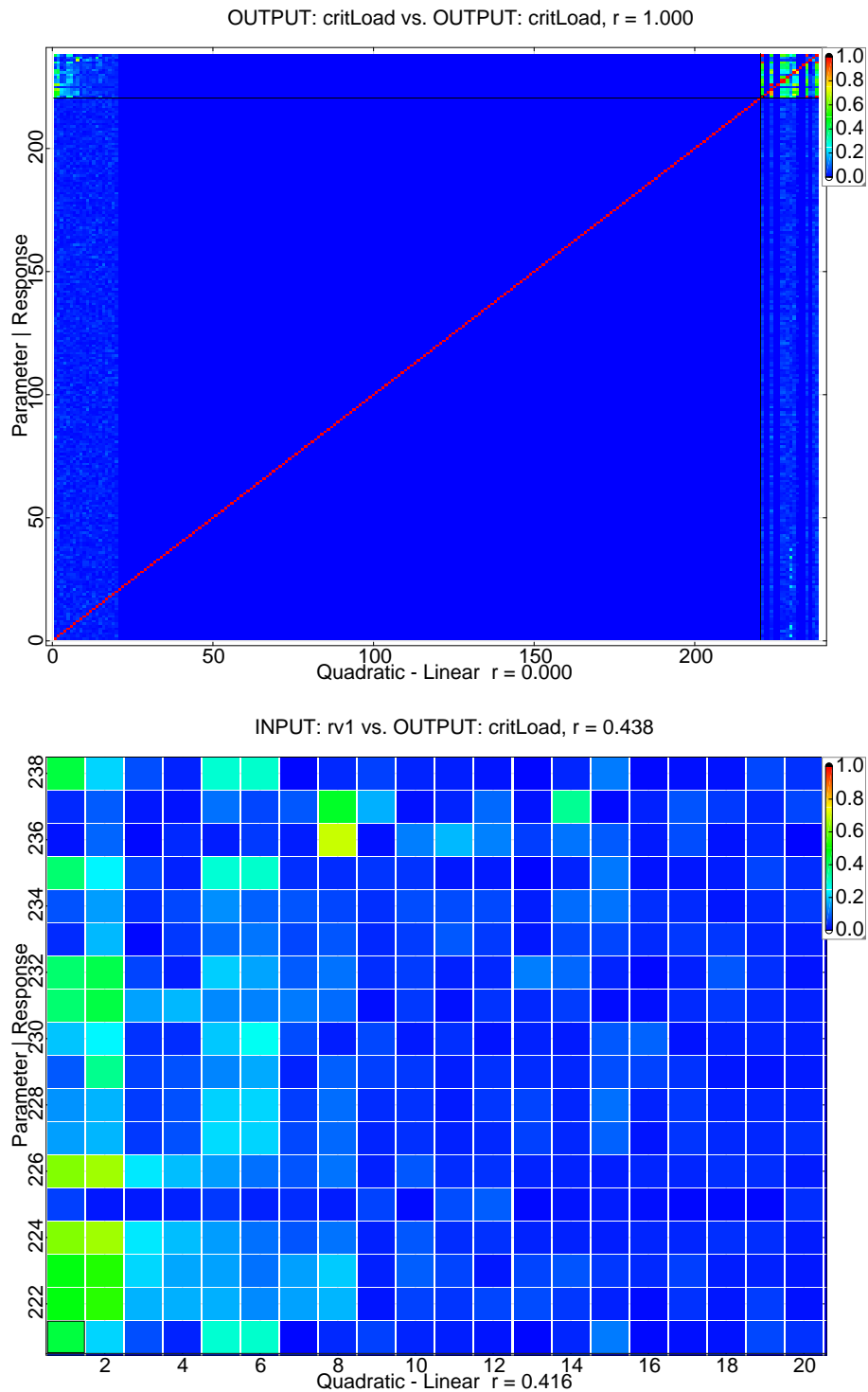


Figure 27: Matrix of quadratic correlation coefficients. Above: full matrix; below: detail. The lowest row shows correlations of the critical load with the first 20 random variables.

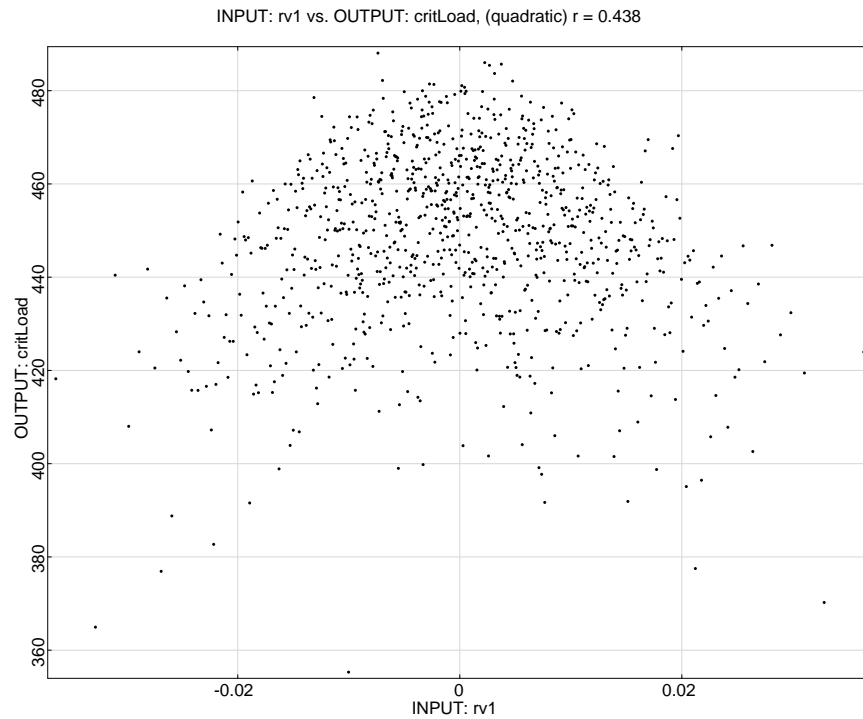


Figure 28: Anthill (or scatter) plot of critical load vs. first random variable.

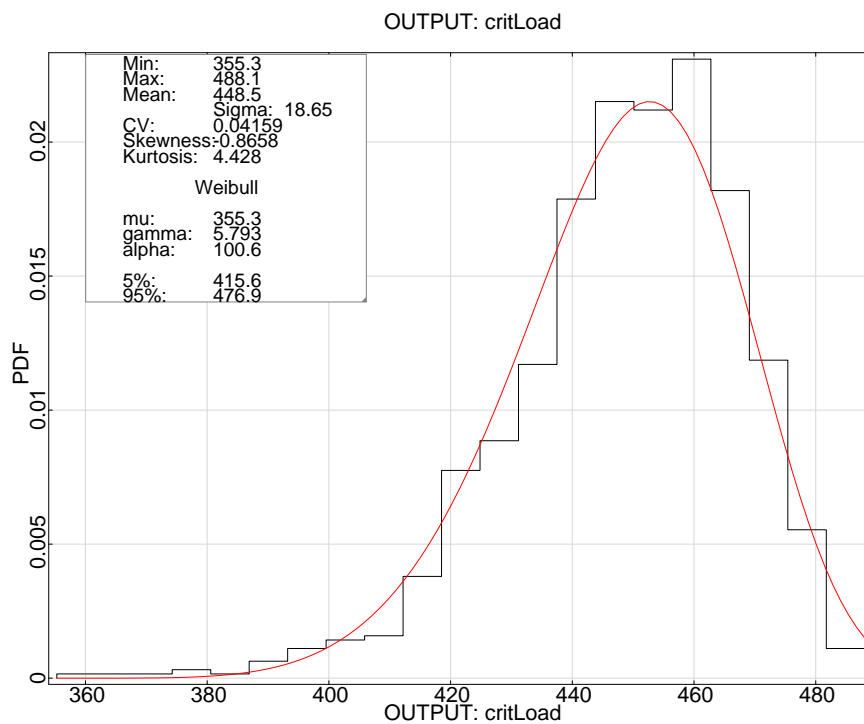


Figure 29: Histogram of the sampled critical load and fitted Weibull distribution.



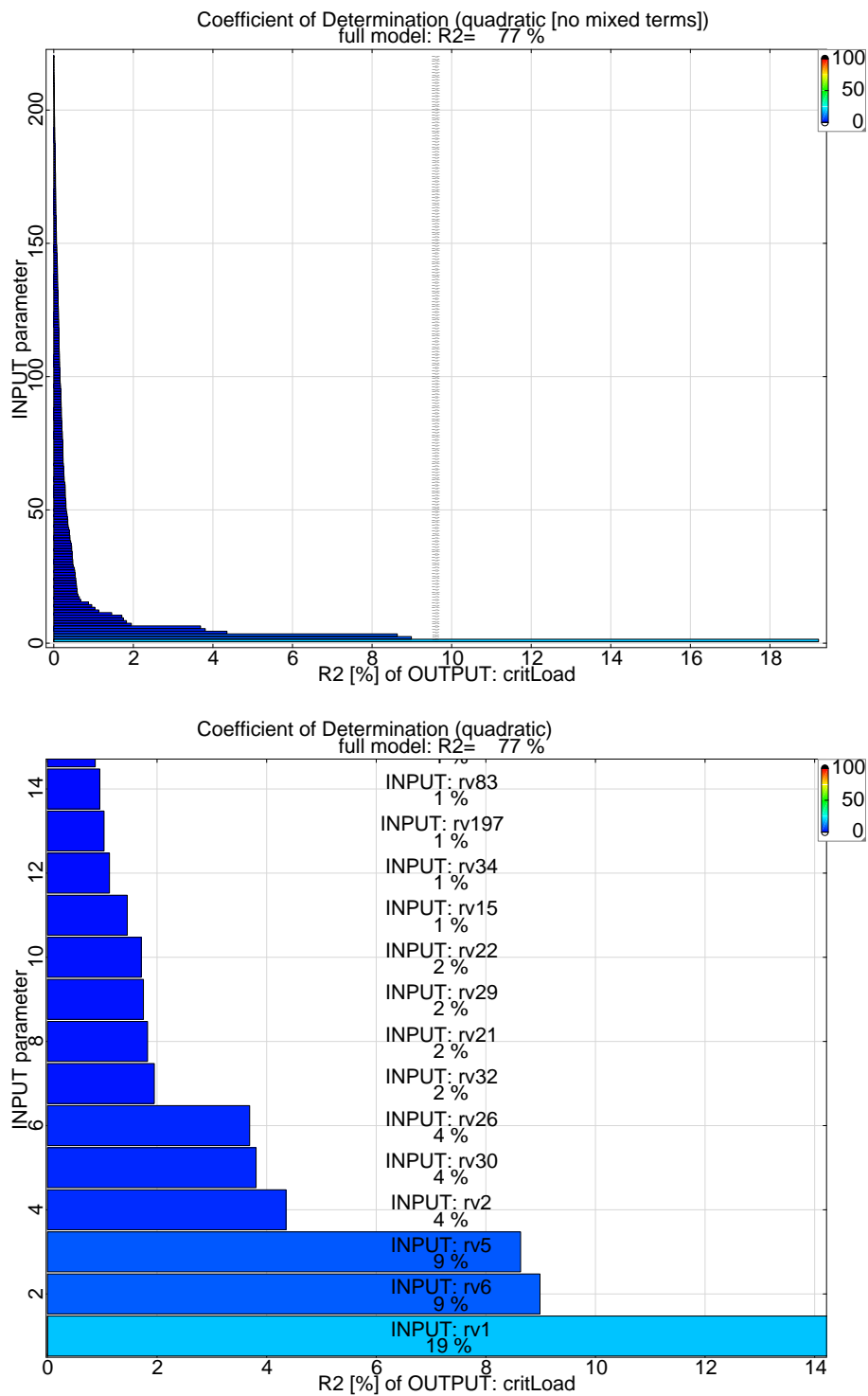


Figure 30: Coefficients of determination of critical load by random variables, for quadratic model. Right: detail.

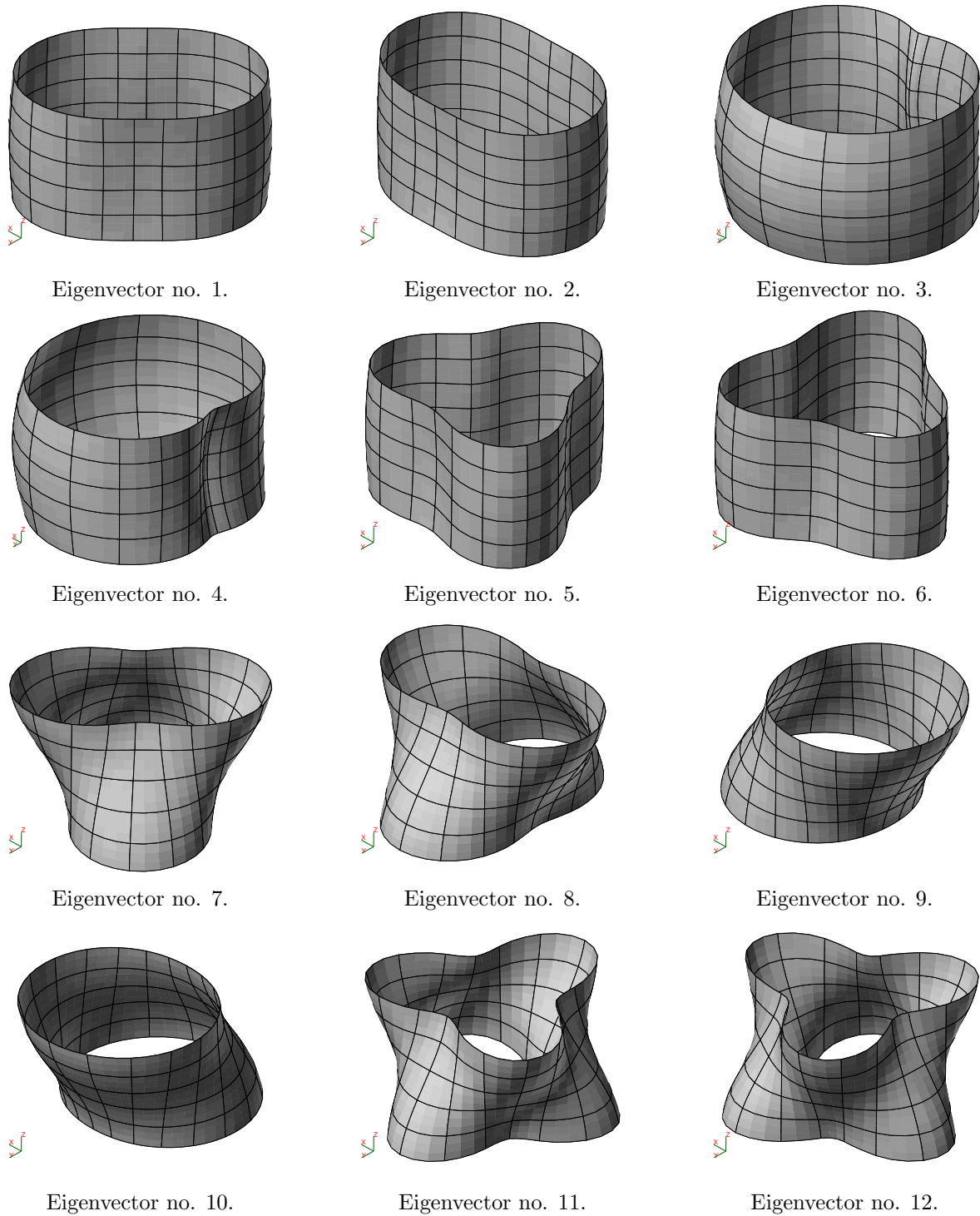
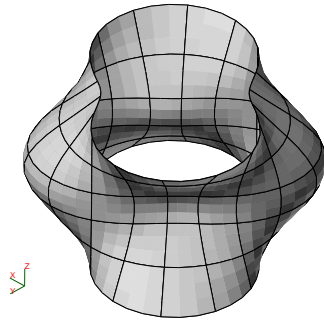
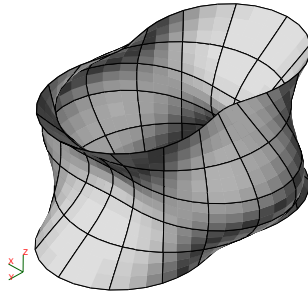


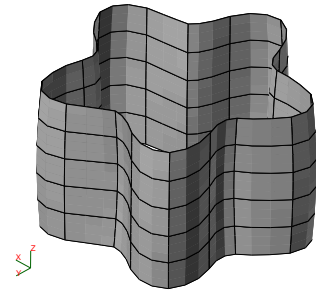
Figure 31: Selected eigenvectors of the covariance matrix.



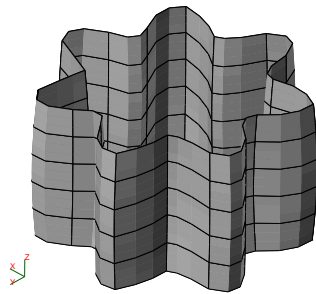
Eigenvector no. 13.



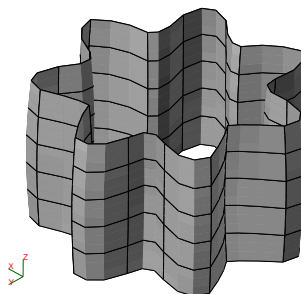
Eigenvector no. 14.



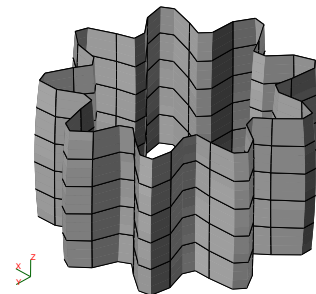
Eigenvector no. 15.



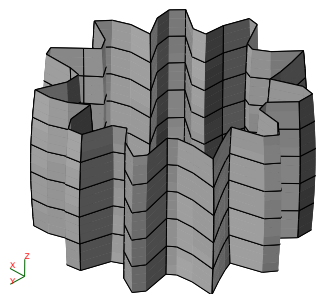
Eigenvector no. 21.



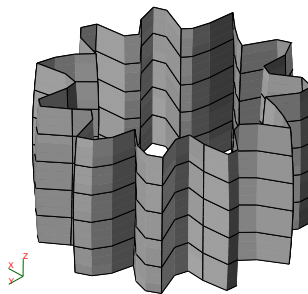
Eigenvector no. 22.



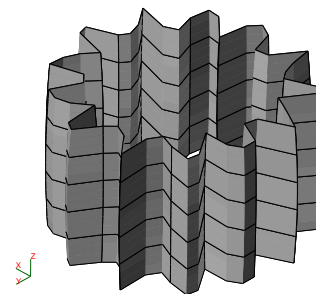
Eigenvector no. 26.



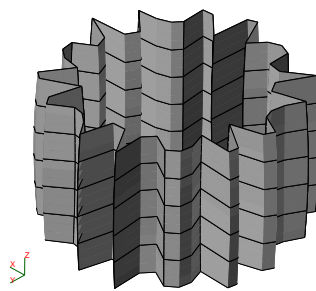
Eigenvector no. 29.



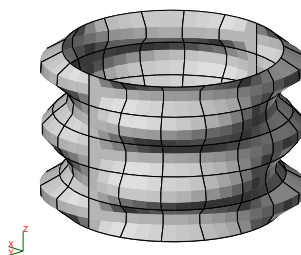
Eigenvector no. 30.



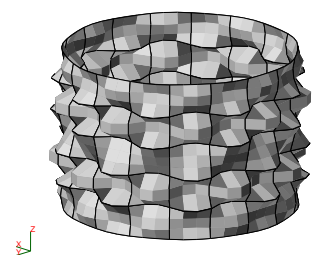
Eigenvector no. 32.



Eigenvector no. 34.



Eigenvector no. 83.



Eigenvector no. 197.

Figure 31 – continued: Selected eigenvectors of the covariance matrix.

**Reliability Analysis** The reliability of the structure is studied by means of a Monte Carlo simulation , with the limit state function as defined by eq. (36). Three variants are computed: as a reference, the full representation of the random field, which employs 396 random variables, is used. Second, the “first 14” variables were selected by the criterion described on page 22 and third, a set of random variables with the “top 14” coefficients of determination, cf. sect. 3.4.2. In each case, a sample with 36000 realizations is generated by Latin Hypercube Sampling. No other variance reduction scheme such as Importance Sampling is applied. Because the random field defines the structural geometry and hence structural behaviour, the limit state function cannot be programmed explicitly, but a linear buckling analysis is carried out for each sample.

The failure probabilities computed with the different sets of variables are listed in table 8. Since Monte Carlo is a statistical method, the so-called statistical error is listed as well. This is the standard deviation of the estimator of the failure probability, which gives information about the confidence of the result. The simulation results with all variables and the “top 14” selection show a good quality. With the “first 14” set of variables, the probability of failure is underestimated by more than a magnitude. This set of random variables is able to represent the random field in good quality, but is not able to model the structural behaviour. The result obtained with the “top 14” selection is close to the reference, although it tends to be lower, too. Obviously, this selection criterion provides a good compromise for both modelling the stochastic and the mechanical problem.

Table 8: Probabilities of failure for different sets of random variables.

No. of random variables Selection criterion	396 none (all)	14 highest $\sigma_{X_i}^2$ (“first 14”)	14 highest $R^2$ (“top 14”)
Prob. of failure $P_f$	$9.7 \cdot 10^{-3}$	$2.8 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$
Statistical error $\sigma_{P_f}$ $\text{cov}(P_f)$	$5.2 \cdot 10^{-4}$ 5 %	$8.8 \cdot 10^{-5}$ 32 %	$3.2 \cdot 10^{-4}$ 9 %

## 4 Concluding Remarks

In this paper different methods for the calculation of failure probabilities are investigated with respect to several parameters as the number of random variables, the probability level, the ability to deal with multiple design points, curvature and the shape of the limit state function and the existence of noise in the response function.

The most robust methods are Monte Carlo simulation and latin hypercube simulation. The main drawback of these methods is, that the variance of the estimator of the probability is quadratically dependent on the probability level, and as a result for low probabilities a large number of samples is required.

A linearization of the limit state function at the design point with FORM (First Order Reliability Method) gives sometimes accurate results and at least an idea about the expected failure probability and for linear problems the exact probability is obtained. FORM requires the least number of samples compared to direct simulation methods, but it is not able to deal with multiple design points and the determination of the design point has to be possible. For certain problems with a noisy response the gradient-based optimizer has failed, and for highly curved limit state function the linear assumption is not reasonable.

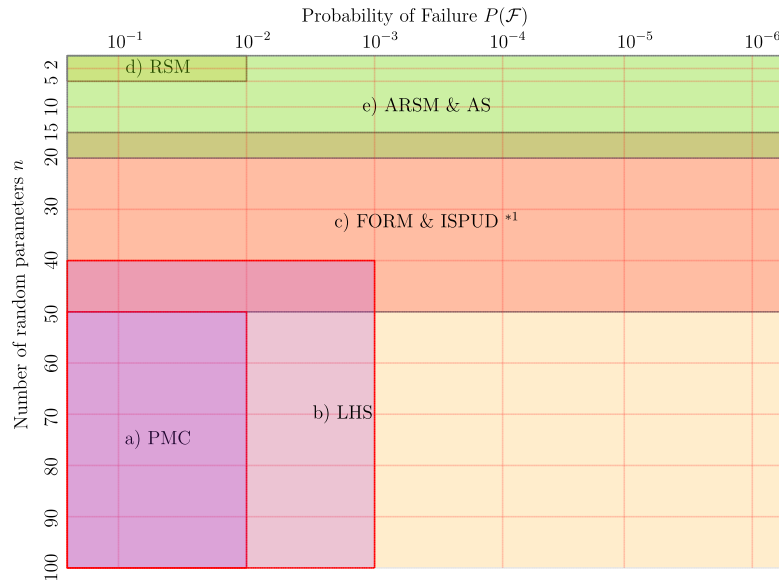


Figure 32: Performance of methods for stochastic analysis. a.) Plain Monte Carlo Simulation (PMC), b) Latin Hypercube Sampling, c.) First Order Reliability Method (FORM) & Importance Sampling using Design Point (ISPUD) (\*<sup>1</sup>only applicable within continuously differentiable response functions), d.) Response Surface Method (RSM), e) Adaptive Response Surface (ARSM) in combination with Adaptive Sampling (AS).

An alternative is the use of importance sampling techniques. The first investigated method is the adaptive sampling approach, where after a first latin hypercube simulation, possibly with adapted variance, the sampling density is determined from the samples of the previous iteration step within the failure domain. The method gives accurate results, if the number of samples within the failure domain was sufficient to accurately determine the correlation, but this number was relatively large, especially in higher dimensions. A second importance sampling technique is ISPUD (Importance Sampling Using Design Point). The sampling is centered near the design point, which is determined in a previous optimization step. The method is a good choice for problems, where the design point can be accurately determined by an optimization (gradient-based optimizer NLPQL).

Methods based on global response surfaces have been tested. It has been observed, that for a small number of random variables ( $n < 5$ ) the global response surface methods are accurate, even for small failure probabilities  $P(\mathcal{F}) \geq 10^{-2}$ . The number of required samples is exponentially increasing with the number of samples, and in higher dimensions ( $n > 10$ ) the response surfaces are not applicable any more.

Finally, a new adaptive response surface method is introduced to analyse the design reliability with high accuracy and efficiency. Whereby the surrogate model is based on an improved moving least square approximation combined with an adaptive design of experiment. In order to obtain a fast simulation procedure on the response surface an adaptive importance sampling concept is developed. In this sense, the proposed method is very robust and efficient for  $n \leq 20$  random parameters and combine the advantages of an adaptive design of experiment, adaptive sampling and efficient response surface methods.

As a conclusion the following proposal for the method of choice and its limitations for a specific problem can be given (see Figure 32):

- Monte Carlo simulation
  - $P(\mathcal{F}) \geq 10^{-1}$ ,  $n$  independent
- Latin hyper cube sampling
  - $P(\mathcal{F}) \geq 10^{-2}$ ,  $n$  independent
- First and second order reliability method
  - $n \leq 50$
  - Continuously differentiable response function
  - Large failure in order high-curved limit state functions
- Importance sampling using design point
  - $n \leq 50$
  - Continuously differentiable structural response function
  - Noisy limit state functions (crash)
- Adaptive importance sampling
  - $P(\mathcal{F}) \leq 10^{-2}$ ,  $n \leq 5$
  - $P(\mathcal{F}) > 10^{-2}$ ,  $n \leq 10$
  - Noisy limit state functions (crash)
  - Non-differentiable structural response function
- Response surface method
  - $P(\mathcal{F}) \geq 10^{-2}$ ,  $n \leq 5$
- Adaptive response surface method
  - $n \leq 20$

In this study, a procedure for calculating the reliability of mechanical systems with random imperfections was developed and tested. Such a complex problem still needs preliminary studies, but is already applicable to realistic problems. Within such preparations, attention must be paid on the analysis of the mechanical problem and choice of the calculation method. A compromise between accuracy and computing time should be found.

The reliability was computed by the Monte Carlo method, with the imperfections modelled as random fields. This may require a huge number of random variables, which in turn may cause numerical difficulties in the computation of the probability of failure. Hence another important task in the preparation phase is a suitable selection of the random variables applied. It is suggested to perform a robustness analysis for this purpose, which requires relatively few additional samples compared to the sample size needed for the reliability analysis itself. The robustness analysis comprises, among other functions, a quadratic regression of the state function by the input random variables. It turned out that for the underlying geometrically non-linear problem, this approach helps to detect the important input variables.

The approach suggested here is called non-parametric, because it depends on stochastic properties defined continuously on the entire structure, but not on geometry parameters. Quite often the randomness of the geometry is introduced by few random structural parameters, such as radius, material thickness etc. These values can be generated by CAD programs. In fact, only variants of the perfect structure are generated that way. It is impossible to consider the random fluctuations over space, which result e.g. from manufacturing tolerances. The presented approach is more effective for this class of problems.

The efficiency of the reliability analysis still can be improved, e.g. by application of Adaptive Response Surface methodology or variance reducing Monte Carlo techniques.

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