Adaptive Response Surfaces for Structural Reliability of Nonlinear Finite Element Structures

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Summary The most general method to solve stochastic problems in structural mechanics is the well established Monte Carlo simulation method. However, the major shortcoming of this approach is its vast need of computational resources (the number of finite elements runs required) which cannot be provided in general situations.

Thus approximations become important which can be based e.g. on the response surface method. Unfortunately, the global approximation schemes widely used in the application of the response surface method can be quite misleading due to the lack of information in certain regions of the random variable space. It is therefore required to avoid such undesirable interpolation errors at reasonable computational effort. The polynomial approximations are not quite flexible. They always need a predefined number of limit state check points in unimportant directions in order to avoid any approximation problems. On this account the maximum number of limit state check points is limited, too.

In this study some new local-global interpolation strategies for the response surface method are proposed. The so-called polyhedral and weighted radii interpolations of the failure surface are intended to provide reasonably accurate estimates of failure probabilities while maintaining computational efficiency. In particular, these response surfaces can be adaptively refined to consistently increase the accuracy of the estimated failure probability. This is achieved by a combination of random search strategies (based on the adaptive sampling and directional sampling approach) as well as deterministic search refinement together with local and global interpolation schemes. The advantage of these methods is the flexibility for the approximation of highly nonlinear limit state functions. This is especially suitable for the reliability analysis of complex nonlinear structures. An arbitrary number of check points even in high local concentration can be used without approximation problems. In this sense, the proposed method is very robust and efficient.

An numerical example from structural analysis under static loading conditions shows the applicability of these concepts. The probabilistic and structural analysis tasks are performed with the SLang software package.

Keywords Monte Carlo simulation; response surface; reliability analysis; nonlinear systems; stochastic mechanics; directional sampling; adaptive sampling

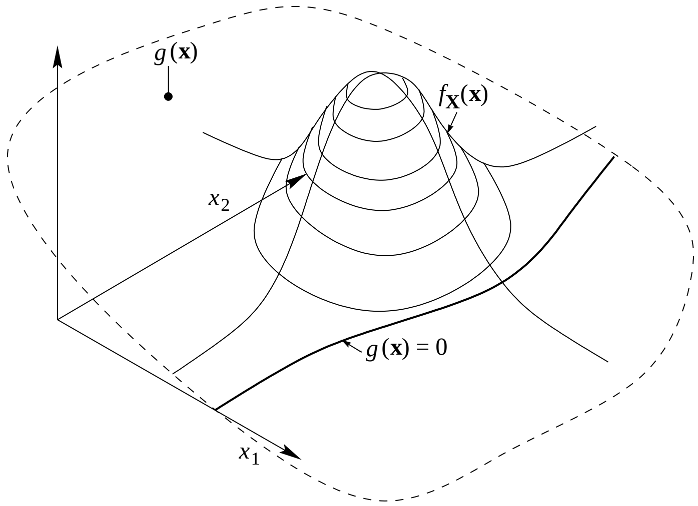

Figure 1: $f_{\mathbf{X}}(\mathbf{x})$: joint probability density function, $g(\mathbf{x})$: structural response function, $g(\mathbf{x}) = 0$: limit state function.

1 Introduction

The structural behavior near the structural failure state is most important in the reliability analysis. The structural design parameters, such as loadings, material parameters and geometry, are the set of basic random variables $\mathbf{X}$ which determine the probabilistic response of structural systems. The failure condition is defined by a deterministic limit state function

$$g(\mathbf{x}) = g(x_1, x_2, \ldots, x_n) \leq 0$$

as shown in Fig. 1. The failure probability of a structural system is given by

$$P(F) = P[\mathbf{X} : g(\mathbf{X}) \leq 0] = \int \overset{n}{\ldots} \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of the basic random variables.

Normally, the response function $g(\mathbf{x})$ of a structural system is described implicitly, e.g. through an algorithmic procedure within finite element analysis. Alternatively, the original structural response function can be approximated by a response surface function $\tilde{g}(\mathbf{x})$ which has polynomial form (Rackwitz (1982); Faravelli (1986); Bucher & Bourgund (1987, 1990)).

A commonly used method for response value approximation is the regression analysis. Usually, the approximation function is a first order or second order polynomial (Box & Draper (1987); Myers (1971)). As an example in the $(n = 2)$-dimensional case, k-responses $(k = 1, ..., m)$ will be approximated using a least square quadratic polynomial in the following form:

$$\tilde{g}_k(\mathbf{x}) = \beta_1 x_{1k} + \beta_2 x_{2k} + \beta_{11} x_{1k}^2 + \beta_{22} x_{2k}^2 + 2\beta_{12} x_{1k} x_{2k} + \epsilon_k \tag{2}$$

Herein the term ϵ_k represents the approximation errors. The approximate coefficients β can be calculated using the least square postulate

$$S = \sum_{k=1}^{m} \epsilon_k^2 = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} \rightarrow \min$$

Additional the limit state function $g(\mathbf{x}) = 0$ can be interpolated by second order polynomials (Bucher *et al.* (1989); Ouypornprasert & Bucher (1988); Bucher *et al.* (1988)).

One of the major advantages of the response surface method lies in its potential to selectively determine the number of structural analyses of the support points. This is especially helpful if some overall

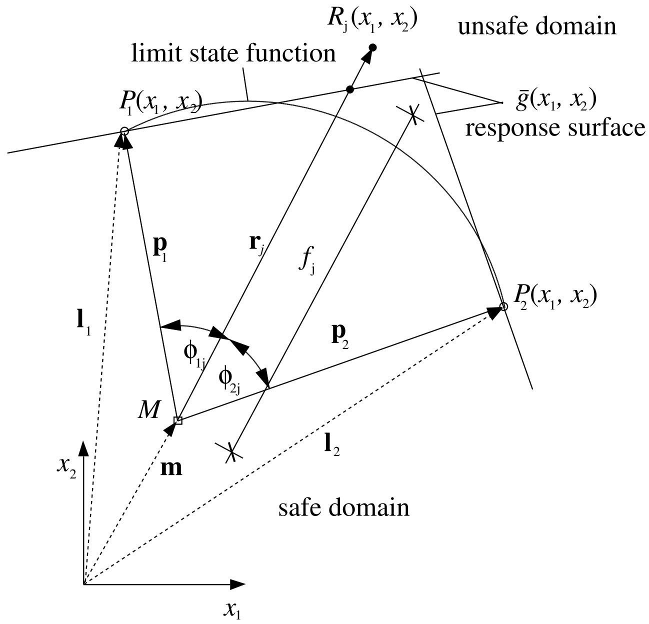

Figure 2: Limit state interpolation using normal hyperplanes.

knowledge on the system behavior - particularly near to the failure region - is a priori available. By such means the computational effort can be substantially reduced.

On the other hand, the global approximation schemes widely used in the application of the response surface method can be quite misleading due to the lack of information in certain regions of the random variable space. It is therefore required to avoid such undesirable interpolation errors at reasonable computational effort.

Standard differentiable (e.g. second order polynomials) approximations are not sufficiently flexible. They always need a predefined number of limit state check points in unimportant regions (or directions) in order to avoid approximation problems in the important region. The maximum number of limit state check points is restricted too.

In the present paper new local or global interpolation strategies for the response surface method are proposed. Lengths and angles of the limit state check point vectors are being used only without any additional geometrical conditions. This is reasonable because, in general, the limit state function between the check points is unknown. Using a combination of adaptive directional sampling these response surfaces can be adaptively refined to consistently increase the accuracy of the response surface function.

2 Polyhedral response surfaces

2.1 Introduction

Non-differentiable polyhedral response surface functions, see e.g. Guan & Melchers (1995, 1997), avoid some problems but need a fairly large number of check points in the case of a closed safe domain. By using a high number of locally concentrated check points approximation problems may appear as well. In the following two local response surface interpolations are presented: normal hyperplane interpolation and a modified secantial hyperplane interpolation (first announcement in Roos *et al.* (1999)).

2.2 Normal hyperplanes

Let the limit state check points $P_i(\mathbf{x})$ be defined by their distance vectors $\mathbf{p}_i$ from a center point M with coordinate vector $\mathbf{m}$. For the subsequent analysis, M must be located in the safe domain, e.g. defined

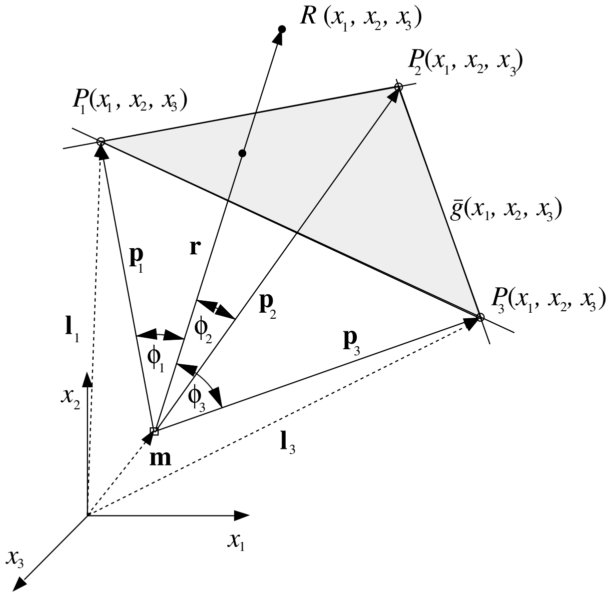

Figure 3: Limit state interpolation using secantial hyperplanes.

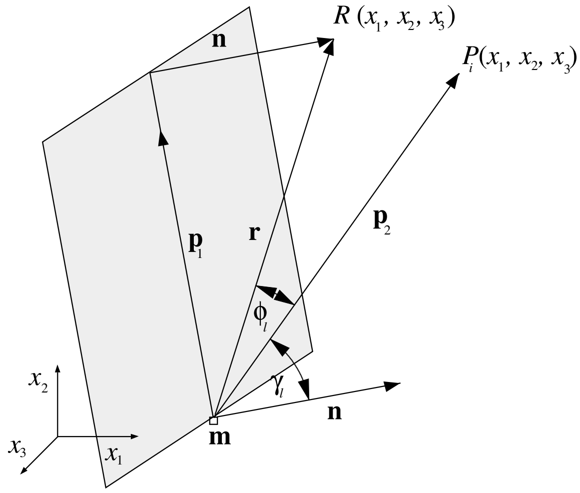

Figure 4: Selection criterion of the check points.

by the expected values of all random variables $\mathbf{m} = \{E[X_1], E[X_2], \ldots, E[X_n]\}^T$:

$$\mathbf{p}_i = \mathbf{l}_i - \mathbf{m}$$

with the limit state check point vectors in Cartesian coordinates $\mathbf{l}_i(\mathbf{x})$. The sampling point $R_j(\mathbf{x})$ is selected from the mean values. Thus the angles between the sampling point j and any limit state check point i are given by

$$\cos\phi_{ij} = \frac{\mathbf{p}_i^T \mathbf{r}_j}{\|\mathbf{p}_i\| \|\mathbf{r}_j\|}$$

Assume that the hyperplane of any limit state point with respect to the mean value is given by the Hessian normal form

$$\mathbf{e}_{p_i}^T \ f_{ij} \mathbf{r}_j = \|\mathbf{p}_i\|$$

then the factors

$$f_j = \left. \frac{\|\mathbf{p}_i\|}{\mathbf{e}_{p_i}^T \ \mathbf{r}_j} \right|_{i: \cos\phi_{ij} \to max}$$

as shown in Fig. 2, give the response surface function

$$\tilde{\mathbf{g}}(\mathbf{x}) = \mathbf{m}(\mathbf{x}) + f_j \mathbf{r}_j(\mathbf{x})$$

in cartesian coordinates. The normal hyperplane type response surfaces as defined here will be star-shaped with respect to the center point M (but not necessarily convex in any case).

2.3 Secantial hyperplanes

An alternative polyhedral function is provided by the following algorithm. For the n dimensional random variable space a n-dimensional hyperplane can be determined by n limit state check points. Such a hyperplane results from the linear combination

$$\tilde{\mathbf{g}}(x_1, \ldots, x_n) = \mathbf{m}(x_1, \ldots, x_n) + \mathbf{p}_1 + \alpha_1(\mathbf{p}_2 - \mathbf{p}_1) + \ldots + \alpha_{n-1}(\mathbf{p}_n - \mathbf{p}_1) \quad (3)$$

as shown in Fig. (3). The equation system

$$[\mathbf{p}_2 - \mathbf{p}_1 \ldots \mathbf{p}_n - \mathbf{p}_1 \;\; -\mathbf{r}]\,[\alpha_1 \ldots \alpha_{n-1}\;\, f]^T = -\mathbf{p}_1$$

leads to the intersection point between the hyperplane and the sampling point vector. Here, the first plane check point should be a vector $\mathbf{p}_i$ with the property $\cos\phi_i \to max$. In order to select the next limit state points $(\mathbf{p}_2, \ldots, \mathbf{p}_n)$ we can use $(\cos(\phi_i) \to max : \cos(\gamma_i) > 0)$ as shown in Fig. 4. This criterion ensures the section of the half space which is of interest. The normal vector n is given by

$$\mathbf{n} = \mathbf{r} - \sum_{l=1}^{n-1} \beta_l \mathbf{p}_l \tag{4}$$

and

$$\mathbf{n}^T \mathbf{p}_m = 0 \tag{5}$$

for all $(l, m = 1 \ldots n-1)$. The equations (4,5) give the following equation system with the maximum size $(n-1)$.

$$\left(\sum_{l=1}^{n-1} \beta_l \mathbf{p}_l\right)^T \mathbf{p}_m = \mathbf{r}^T \mathbf{p}_m$$

With the coordinates β_l of the nadir we obtain the normal vector using the equations (4) and (5). In case that the coefficients matrix $\mathbf{P} = \mathbf{p}_l^T \mathbf{p}_m$ is singular we can simply determine the factor f by

$$f\mathbf{r} = \sum_{l=1}^{n-1} \beta_l \mathbf{p}_l \tag{6}$$

because that means $\mathbf{n} = \mathbf{0}$. The solution of equation (6) is given by the linear least squares problem

$$\|f\mathbf{r} - \sum_{l=1}^{n-1} \beta_l \mathbf{p}_l\| \to min$$

Finally, we have to check these points which were found for computing the hyperplane with the conditions

$$\alpha_l \geq 0; \sum_l \alpha_l \leq 0 \;\forall\, (l = 1 \ldots n-1)$$

That means that the intersection point of the sampling vector with the hyperplane is inside the plane. Otherwise we have to interchange the n-th limit state check point with the closest point on the correct half space.

2.4 Weighted radii response surfaces

A global or local response surface method is the weighted radii interpolation. Let the limit state check points $P_i(\mathbf{x})$ be applied to mean value coordinates on the expected values of all random variables $E[X_1], E[X_2], \ldots, E[X_n]$ by

$$\mathbf{p}_i = \mathbf{l}_i - \mathbf{m}$$

with the limit state check point vector in cartesian coordinates $\mathbf{l}_i(\mathbf{x})$ and the mean value vectors $\mathbf{m}(\mathbf{x})$. The sampling point $R_j(\mathbf{x})$ is selected from the means with $\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{m}$. Thus the angles between the sampling point j and any limit state check point i is given by

$$\cos\phi_{ij} = \frac{\mathbf{p}_i^T \mathbf{r}_j}{\|\mathbf{p}_i\| \|\mathbf{r}_j\|} \qquad 0 \leq \phi_{ij} \leq \pi$$

Assume that the weights of any sampling point are given by

$$w_{ij} = \frac{1}{\phi_{ij}} \tag{7}$$

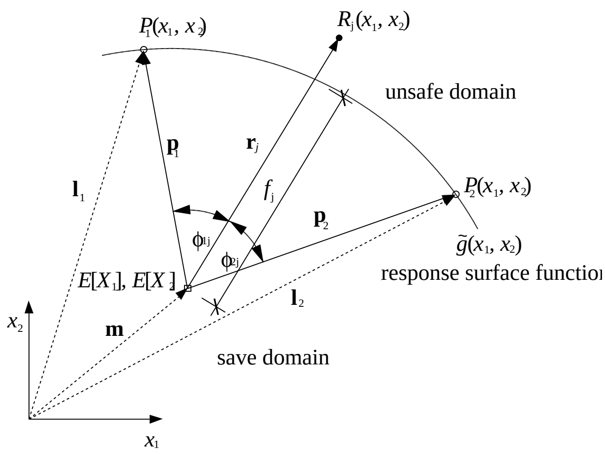

Figure 5: Limit state interpolation using weighted radii.

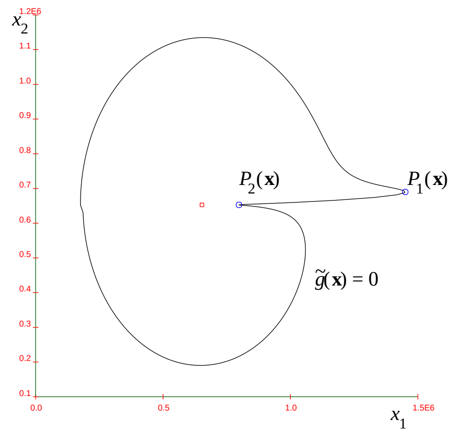

Figure 6: Interpolation without difficulties in near of discontinuities.

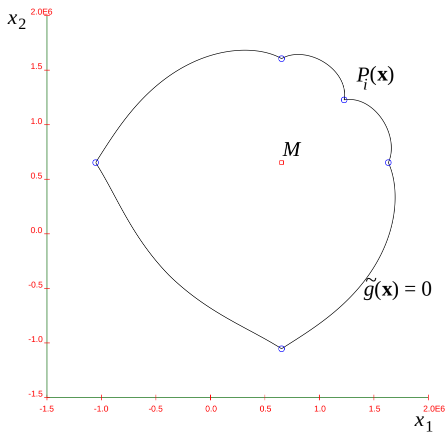

Figure 7: Response surface by using linear weights.

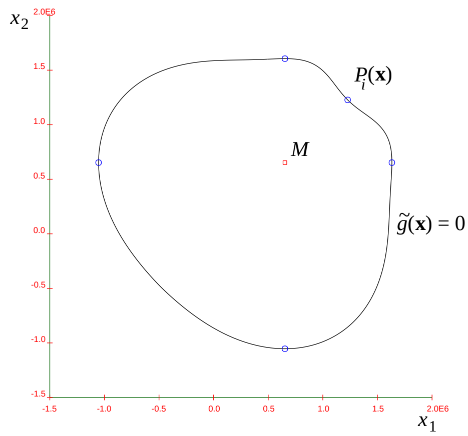

Figure 8: Response surface by using nonlinear weights.

then the factors

$$f_j = \frac{\sum_i \|\mathbf{p}_i\| w_{ij}}{\sum_i w_{ij}}$$

as shown in Fig. 5, give the response surface function

$$\tilde{\mathbf{g}}(\mathbf{x}) = \mathbf{m}(\mathbf{x}) + f \frac{\mathbf{r}(\mathbf{x})}{\|\mathbf{r}(\mathbf{x})\|} = \mathbf{0}$$

on the cartesian coordinates. The simple assumption (7) generates a closed response surface function. To eliminate the numerical discontinuity near $\phi_{ij} = 0$ we can introduce a small value ϵ in the equation (7)

$$w_{ij} = \frac{1}{\phi_{ij} + \epsilon}$$

The response surface function for some limit state check points using linear weights is shown in Fig. 7. Introducing nonlinear weights such as

$$w_{ij} = \left(\frac{1}{\phi_{ij} + \epsilon} \right)^2$$

will form a differentiable function in the supporting points, as shown in Fig. 8. Using the presented interpolation, there are no difficulties in near of discontinuities, as shown in Fig. 6, in contrast to polyhedral and polynomial response surface functions.

3 Adaptive directional sampling

The unit vectors simulation can be shifted to the dominant areas of the standard Gaussian space. Any point $\mathbf{u}$ in this space can be written in polar coordinates

$$\mathbf{u} = r\mathbf{a}$$

Herein

$$r = \|\mathbf{u}\|$$

is the vector length of $\mathbf{u}$, measured relative to the point of origin and

$$\mathbf{a} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

is a unit vector who determines the direction. Now we introduce a specific simulation density function $h_{\mathbf{Y}}(\mathbf{a})$ for the directions $\mathbf{a}$. The postulation that the first and second moments are equal to the statistical moments of the failure samples (Bucher (1988))

$$E[\mathbf{Y}] = E[\mathbf{A}|g(r^*(\mathbf{a})\ \mathbf{a}) \le 0]$$
$$E[(\mathbf{Y}-\bar{\mathbf{Y}})(\mathbf{Y}-\bar{\mathbf{Y}})^T] = E[(\mathbf{A}-\bar{\mathbf{A}})(\mathbf{A}-\bar{\mathbf{A}})^T|g(r^*(\mathbf{a})\ \mathbf{a}) \le 0]$$

produces an approximation of the "optimal" sampling density. Instead of it can be used

$$E[\mathbf{Y}] = \frac{\sum_{i=1}^N \mathbf{a}_i\ w_i}{\sum_{i=1}^N w_i}$$
$$E[(\mathbf{Y}-\bar{\mathbf{Y}})(\mathbf{Y}-\bar{\mathbf{Y}})^T] = \frac{\sum_{i=1}^N w_i(\mathbf{a}_i-\bar{\mathbf{a}})(\mathbf{a}_i-\bar{\mathbf{a}})^T}{\sum_{i=1}^N w_i}$$

in order to determination of the sampling density, as defined in Kijawatworawet (1991); Kijawatworawet *et al.* (1998). Whereas the weights w_i are defined by

$$w_i = \int_{r_i^*(\mathbf{a})}^{\infty} r_i^{n-1} \exp\left(-\frac{r_i^2}{2}\right)\ dr_i$$

4 Numerical example – buckling collapse of a parabolic shell

4.1 Mechanical system and random variables

An example serves to demonstrate the applicability of the adaptive response surface method using the polyhedral and weighted radii interpolation and to discuss some properties of the method. The computational probabilistic and structural analysis tasks were implemented in the software package SLang Bucher *et al.* (1995). SLang integrates geometrically and physically nonlinear finite elements as well as different sampling strategies such as importance and adaptive sampling. Herein the mechanical system is a parabolic shell, as shown in Fig. 9, subjected to horizontal and vertical loads, $13X_1$ and $40X_2$. The constitutive relation of the shell elements is von Mises plasticity without hardening. The structure is modeled with 120 geometrically nonlinear shell elements (SHELL93, Young's modulus $2.1E11\ N/m^2$, yield stress $2.4E8\ N/m^2$, Poisson's ratio 0.3). It is assumed that the applied vertical and horizontal loads as well as the shell thickness are independent Gaussian random variables with statistics given in Table 1.

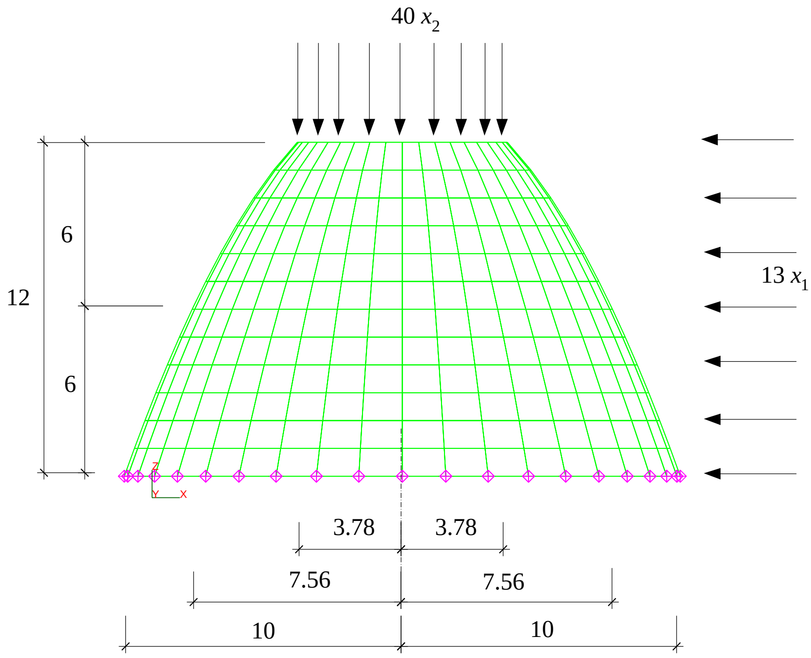

Figure 9: Finite element discretization subjected to loads.

Table 1: Statistical data for loads and thickness

	Mean	Std.Dev	Type
Horizontal Load	$E[X_1] = 2.8e + 07$	$\sigma_{X_1} = 3e + 06$	normal
Vertical Load	$E[X_2] = 2e + 07$	$\sigma_{X_2} = 1e + 06$	normal
Thickness	$E[X_3] = 0.5$	$\sigma_{X_3} = 0.005$	normal

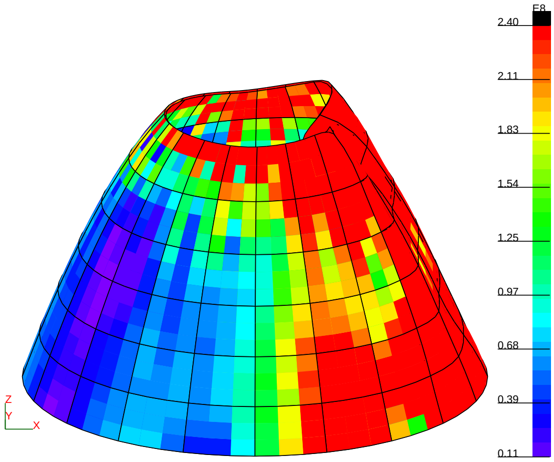

Figure 10: von-Mises stresses and displacements for one chosen limit state point.

4.2 Nonlinear finite element limit state analysis

We obtain the physically nonlinear finite element reference result by using directional sampling (Deák (1980); Bjerager (1988); Ditlevsen & Bjerager (1988)) with 700 support points. Limit state points were determined by means of incremental analysis. The loads are incremented until collapse of the structure. The failure in equilibrium iteration was used to determine collapse of the structure. Numerically, this is defined by non-convergence of the Newton-Raphson iteration or the singularity of the tangential stiffness matrix. The distribution of the von-Mises stress at collapse under equal horizontal and vertical load is shown in Fig.10. In obtaining these result, the loads were incremented proportionally up to collapse of the structure. Hence no path dependence was considered in the analysis. A set of 700 failure points was calculated using directional sampling as shown in Fig. 11. For the statistical data as given in Table 1 the failure probability is calculated as $\bar{P}(F) = 6.2357 \cdot 10^{-6}$ with a statistical error (standard deviation) of $\sigma^2_{\bar{P}(F)} = 9.5434 \cdot 10^{-7}$.

In a first step, the random directions required for directional sampling and the response surface strategy are simulated in a master process. These values are then distributed among different SLang slave processes which are launched in parallel. Finally the master process evaluates the data collected by the slave processes and determines the strategy for the next run. The analysis was carried out in less than 10 hours on an SGI Origin 2000 with 14 processors.

4.3 Adaptive response surface interpolation

Starting from a sample of unit direction vectors we obtain the critical distances on the limit state. A sampling density can be derived and the second run results additional limit state points. Now the new response surfaces can use all of the limit state points and additionally any support points, for example in direction of the axis. Fig. 13 shows $N = 42$ limit state points using random search directions. The search direction simulation is adaptive shifted to the dominant areas of the standard Gaussian space. The additional limit state points are shown in Fig. 14 and 15. The failure probability for arbitrary structures exhibiting nonlinear behavior can be calculated by Monte Carlo simulation. In order to reduce the sample size, an importance sampling concept is used, which automatically adapts the simulation densities in three subsequent simulation runs (Bucher (1988)). Fig. 16 to 21 show the response surfaces using adaptive response surfaces using the polyhedral and weighted radii and first run and second adaptation of the importance sampling. We obtain following convergence of the failure probabilities in dependence on the number N of points on the limit state surface, as shown in Fig. 22. The weighted radii interpolation starts from the polynomial result (Bucher *et al.* (1989)). The slow convergence of the secantial hyperplane method is not surprising, because it represents a consistently safe interpolation. Hence with 28 support points we obtain quite accurate results of the proposed adaptive response surfaces using weighted radii.

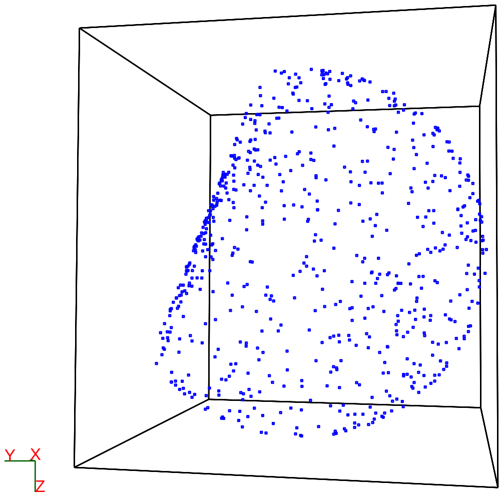

Figure 11: 700 limit state points in the Gaussian space.

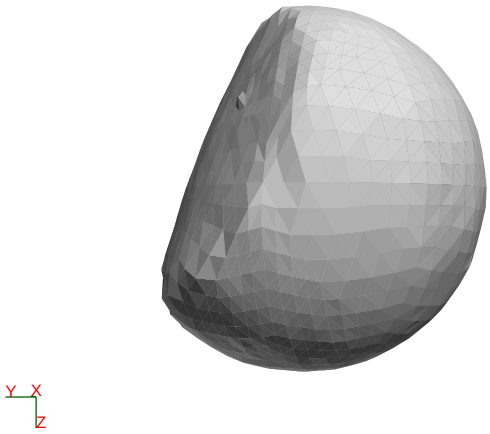

Figure 12: Limit state function in the Gaussian space.

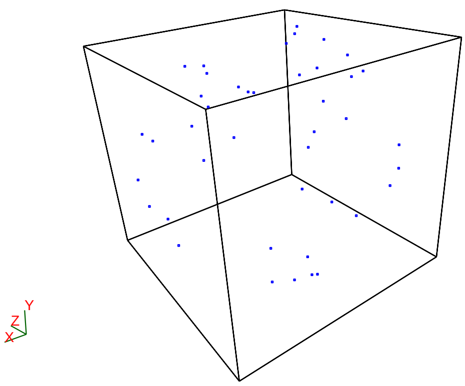

Figure 13: 42 limit state points using random search directions in standard Gaussian space.

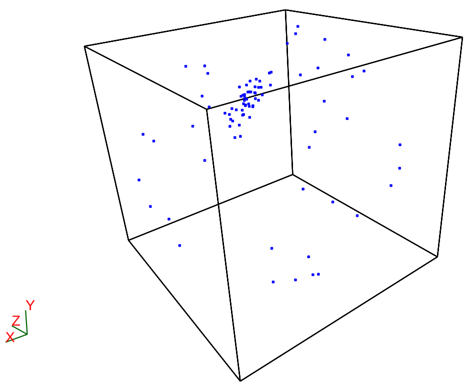

Figure 14: 42 limit state points using random search directions and 42 additional limit state points in standard Gaussian space.

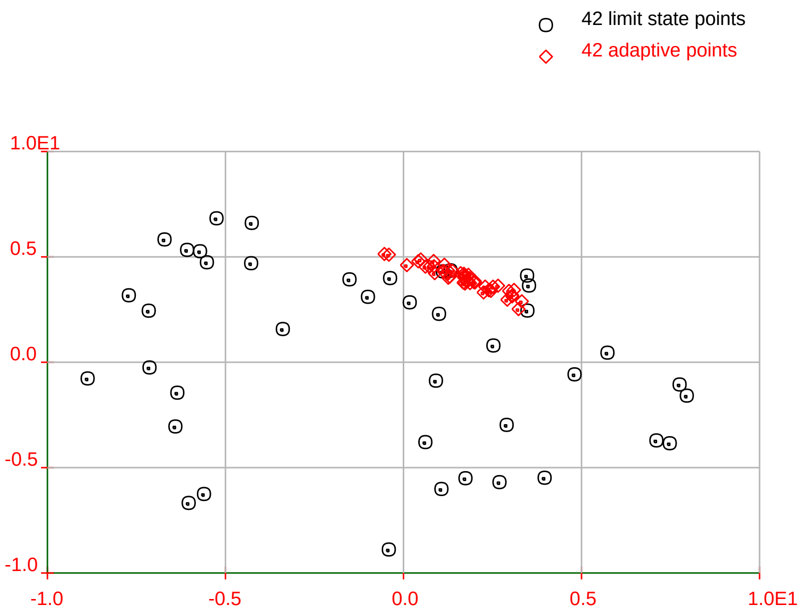

Figure 15: 42 limit state points using random search directions and 42 additional limit state points in standard Gaussian u_1-u_2-plane.

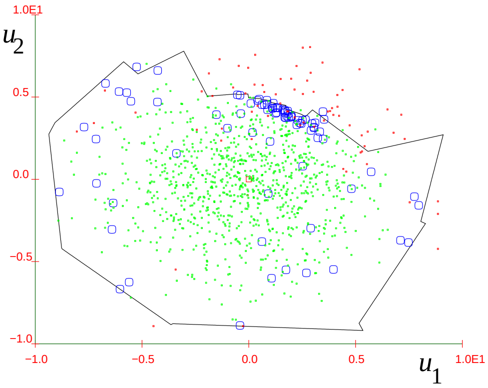

Figure 16: u_1-u_2-plane, interpolation using normal hyperplanes, first run of the importance sampling.

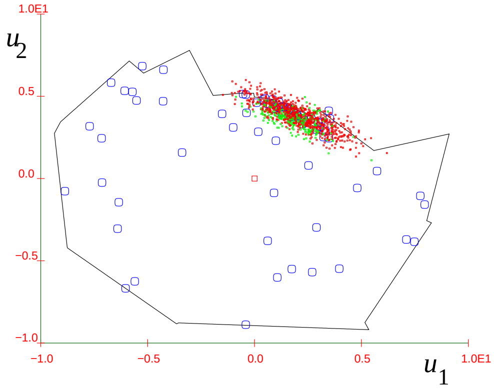

Figure 17: u_1-u_2-plane, interpolation using normal hyperplanes, second adaptation of the importance sampling.

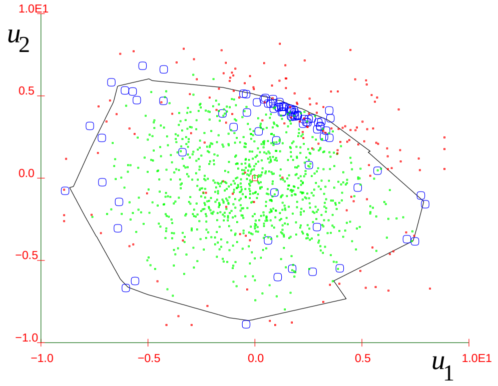

Figure 18: u_1-u_2-plane, interpolation using secantial hyperplanes, first run of the importance sampling.

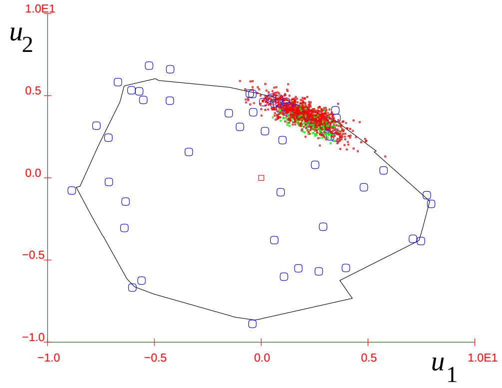

Figure 19: u_1-u_2-plane, interpolation using secantial hyperplanes, second adaptation of the importance sampling.

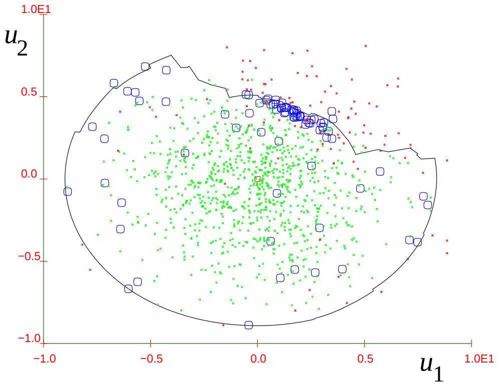

Figure 20: u_1-u_2-plane, interpolation using weighted radii, first run of the importance sampling.

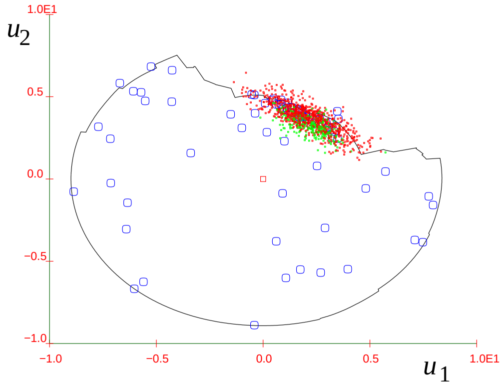

Figure 21: u_1-u_2-plane, interpolation using weighted radii, second adaptation of the importance sampling.

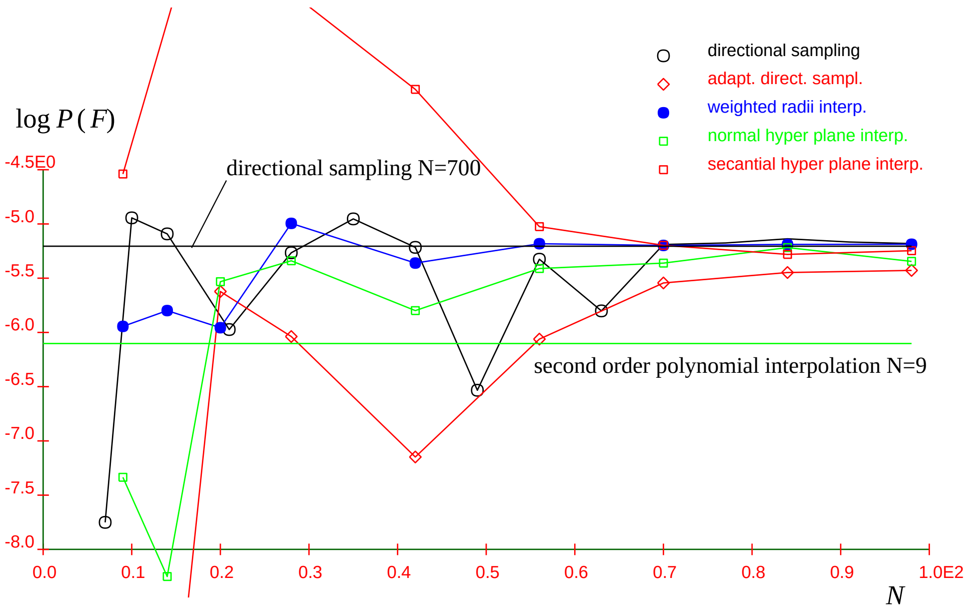

Figure 22: Convergence of the failure probabilities in dependence on the number N of points on the limit state surface.

5 Concluding Remarks

The proposed methods are suitable for computing the reliability of complex structures. The main advantage of these methods is their flexibility for the interpolation of highly nonlinear limit state functions. In particular, these response surfaces can be adaptively refined to consistently increase the accuracy of the estimated failure probability. In this sense, the proposed methods are very robust and combine the advantages of adaptive directional sampling and efficient response surface methods. It should be mentioned that application of this type of response surface should preferably be done in standard Gaussian space.

The response surface interpolation using weighted radii used lengths and angles of the limit state check point vectors only and not any additional geometrical conditions. This is reasonable because, in general, the limit state progression between the check points is unknown. When we have one check point only we obtain a circular safe domain - the most rational assumption. The nonlinear weights produce smooth and continuous functions. We can use any many as you like check points as well local concentrative without approximation problems.

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