Efficient Methods for Robust Design and Optimization
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Basic concepts for robustness evaluation using stochastic analysis

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1 Introduction

During the virtual design process, structural optimization typically aims at high performance levels for a clearly specified set of conditions. Unfortunately, this goal can usually be achieved only by a trade-off involving reduced robustness of the design. This becomes visible as a high sensitivity with respect to unforeseen situations or unavoidable manufacturing tolerances. In order to prevent structural failure due to loss of robustness it is therefore desirable to incorporate a suitable measure of robustness into the optimization process. This can be achieved by introducing additional constraint conditions or appropriate modifications of the objective function. As there are several possible approaches to the notion of "uncertainty", robust design optimization can be based o different mathematical models of uncertainty (see e.g. [1]). Well-known examples are probability theory (involving stochastics) or fuzzy set theory. In the following, focus will be put on stochastic design optimization. A schematic sketch of the approach is given in Fig.1.
An example for a probability-oriented design concept is *reliability-based optimization* which is based on the notion of the failure probability. This is most appropriate for high-risk structures such as e.g. power-generating facilites. Alternatively, more simple stochastic measures such as variances or standard deviations might be appropriate for the design of low-risk structural elements.
Uncertainties in the optimization process can be attributed to three major sources as shown in Fig. 2 These sources of uncertainties or stochastic scatter are

- Uncertainty of design variables. This means that the manufacturing process is unable to achieve the design precisely. The magnitude of such uncertainty depends to a large extent on the quality control of the manufacturing process.

- Uncertainty in the objective function. This means that some parameters affecting the structural performance are beyond the control of the designer. These uncertainties may be reduced by a stringent specification of operating conditions. This may be possible for mechanical structures, but is typically not feasible for civil structures subjected to environmental loading such as earthquakes or severe storms which cannot be controlled.

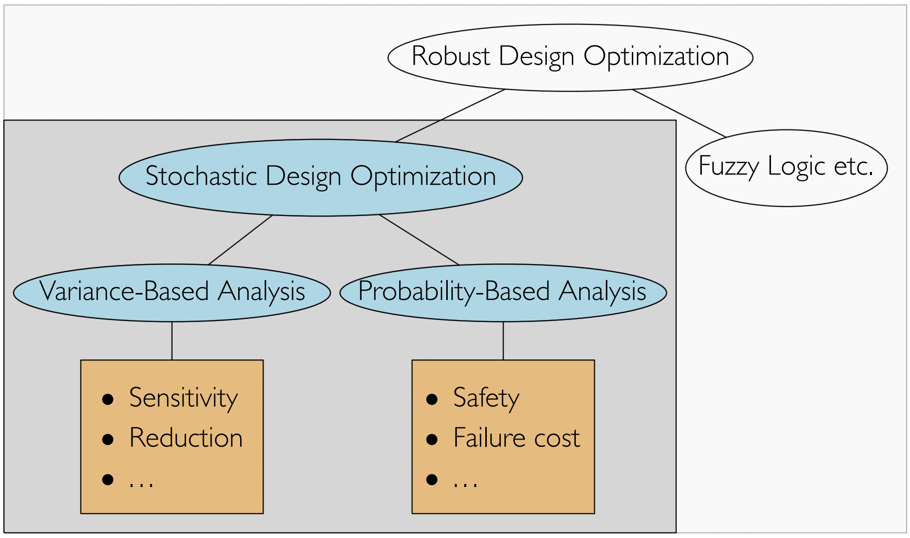


Figure 1: Robustness analysis in the design optimization process

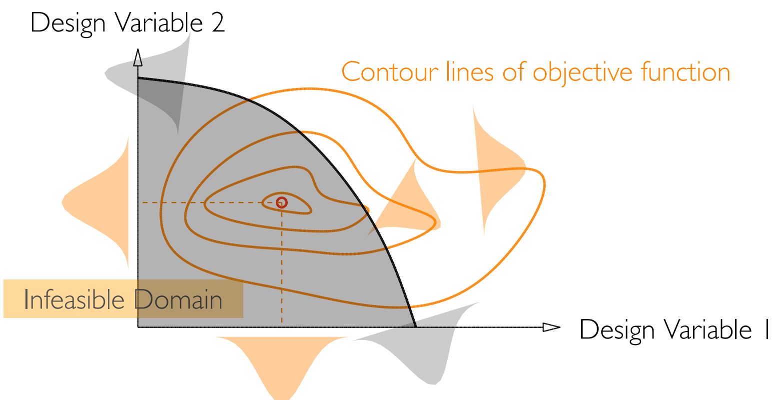


Figure 2: Sources of uncertainty in optimization

- Uncertainty of the feasible domain. This means that the admissibility of a particular design (such as its safety or serviceability) cannot be determined deterministically. Such problems are at the core of probability-based design of structures.

A thorough review of optimization in the context of stochastic mechanics is given e.g. by [3].

2 Stochastic modelling

Probability in the mathematical sense is defined as a positive measure (between 0 and 1) associated with an event in probability space. For most physical phenomena this event is suitably defined by the occurrence of a real-valued random value X which is smaller than a prescribed, deterministic value x. The probability associated with this event is called *probability distribution function* (or, equivalently *cumulative distribution function*, cdf):

$$F_X(x) = P[X < x] \tag{1}$$

Differentiation of $F_X(x)$ with respect to x yields the so-called *probability density function* (pdf):

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{2}$$

A qualitative representation of these functions is given in Fig. 3.

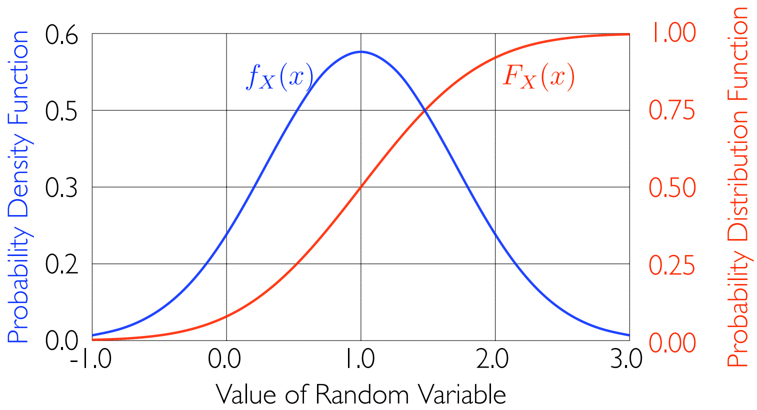


Figure 3: Schematic sketch of probability distribution and probability density functions.

In many cases it is convenient to characterize random variables in terms of expected values rather than probability density functions. Special cases of expected values are the *mean value* $\bar{X}$:

$$\bar{X} = \mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{3}$$

and the *variance* σ_X^2 of a random variable:

$$\sigma_X^2 = \mathbf{E}[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx \tag{4}$$

The positive square root of the variance σ_X is called *standard deviation*. For variables with non-zero mean value ($\bar{X} \neq 0$) it is useful to define the dimension-less coefficient of variation

$$V_X = \frac{\sigma_X}{\bar{X}} \tag{5}$$

A description of random variables in terms of mean value and standard deviation is sometimes called "second moment representation". Note that the mathematical expectations as defined here are so-called *ensemble averages*, i.e. averages over all possible realizations.
Due to its simplicity, the so-called Gaussian or normal distribution is frequently used. A random variable X is *normally distributed*, if its probability density function is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right]; \quad -\infty < x < \infty \tag{6}$$

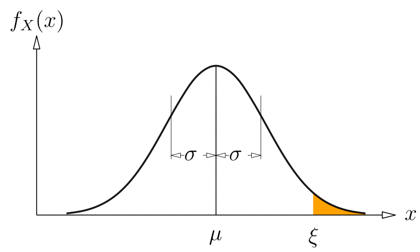
Here $\bar{X}$ is the mean value, and σ_X is the standard deviation. The distribution function $F_X(x)$ is described by the normal integral $\Phi(.)$:

$$F_X(x) = \Phi\left(\frac{x-\bar{X}}{\sigma_X}\right) \tag{7}$$

in which

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{u^2}{2}\right) du \tag{8}$$

This integral is not solvable in closed form, however tables and convenient numerical approximations exist. The use of the Gaussian distribution is frequently motivated by the central limit theorem which states that an additive superposition of independent random effects tends asymptotically to the Gaussian distribution.



$$P_\xi = P[X \geq \xi]$$

ξ	μ	$\mu+\sigma$	$\mu+2\sigma$	$\mu+3\sigma$	$\mu+4\sigma$
P_ξ	$5 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$2.3 \cdot 10^{-2}$	$1.4 \cdot 10^{-3}$	$3.2 \cdot 10^{-5}$

Figure 4: Gaussian (normal) probability density function and probabilites of exceeding threshold values ξ

A random variable X is *log-normally distributed*, if its pdf is:

$$f_X(x) = \frac{1}{x\sqrt{2\pi}s} \exp\left[-\frac{(\log\frac{x}{\mu})^2)}{2s^2}\right]; \quad 0 \leq x < \infty \tag{9}$$

and its distribution function is given by

$$F_X(x) = \Phi(\frac{\log \frac{x}{\mu}}{s}) \tag{10}$$

In these equations, the parameters μ and s are related to the mean value and the standard deviation as follows:

$$\mu = \bar{X} \exp(-\frac{s^2}{2}); \quad s = \sqrt{\ln(\frac{\sigma_X^2}{\bar{X}^2} + 1)} \tag{11}$$

Two random variables with $\bar{X} = 1.0$ and $\sigma_X = 0.5$ having different distribution types are shown in Fig. 5. It is clearly seen that the log-normal density function is non-symmetric and does not allow negative values. Another important difference lies in the fact that the probability

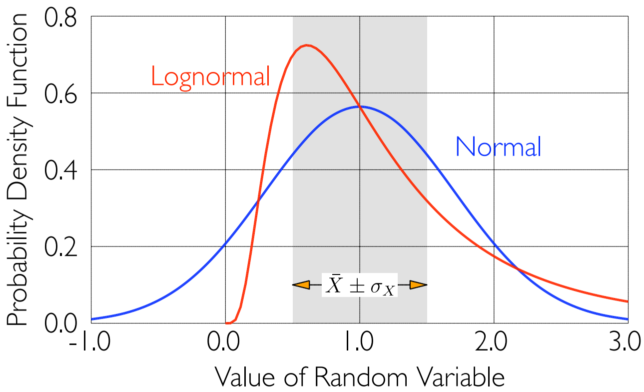


Figure 5: Normal and log-normal probability density functions.

of exceeding certain threshold levels ξ is significantly influenced by the type of probability distribution. For a normal distribution, the probability of exceeding a level $\xi = 3$ corresponding to the mean value plus 4 standard deviations is $3.2 \cdot 10^{-5}$ while in the case of a lognormal distribution the same threshold has an exceedance probability of 0.083. In order to achieve the same exceedance probability as in the Gaussian case, the threshold level must be set to $\xi = 7.39$, which is the mean value plus 12 standard deviations.
In many applications a large number of random variables occur together. It is conceptually helpful to assemble all these random variables X_k; $k = 1 \ldots n$ into a *random vector* $\mathbf{X}$:

$$\mathbf{X} = [X_1, X_2, \ldots X_n]^T \tag{12}$$

For this vector, expected values can be defined in terms of expected values for all of its components:

Mean value vector

$$\bar{\mathbf{X}} = \mathsf{E}[\mathbf{X}] = [\bar{X}_1, \bar{X}_2, \ldots \bar{X}_n]^T \tag{13}$$

Covariance matrix

$$\mathsf{E}[(\mathbf{X}-\bar{\mathbf{X}})(\mathbf{X}-\bar{\mathbf{X}})^T] = \mathbf{C_{XX}} \tag{14}$$

The dimensionless quantity

$$\rho_{ik} = \frac{\mathsf{E}[(X_i-\bar{X}_i)(X_k-\bar{X}_k)]}{\sigma_{X_i}\sigma_{X_k}} \tag{15}$$

is called *coefficient of correlation*. Its value is bounded in the interval $[-1,\ 1]$.

3 Variance-based analysis

3.1 General remarks

In order to obtain meaningful correlations between the input and output variables it is essential to precisely capture the input correlations in the simulated values. Monte-Carlo based methods use digital generation of pseudo-random numbers to produce artificial sample values for the input variables. The quality of these numbers can be measured in terms of their statistical properties. For the case of two random variables X_1 and X_2, Monte Carlo methods produce sequences of numbers $X_1^k, X_2^k, k = 1 \dots N$ in such a way that the prescribed statistics as estimated from these samples match the prescribed statistics as closely as possible. Typically, plain Mont-Carlo methods are fairly well able to represent individual statistics of the random variables. At small sample sizes N, however, the prescribed correlation structure may be rather heavily distorted. Significant improvement can be made by utilizing the Latin Hypercube sampling method [2]. Unfortunately, many real-world structural problem are so large that only a small number of samples can be accepted.

3.2 Correlation Statistics

Assume that we want to estimate a matrix of correlation coefficients of m variables from N samples. This matrix has $M = m \cdot (m-1)/2$ different entries in addition to m unit elements on the main diagonal. The confidence intervals for the estimated coefficients of correlation ρ_{ij} are

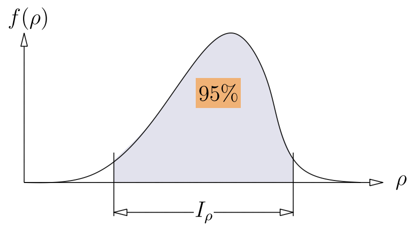


Figure 6: Confidence interval for estimated coefficients of correlation ρ

computed based on the Fisher's z-transformation. The interval for a significance level of α (i.e. a confidence level of $1-\alpha$) is given by

$$[\tanh(z_{ij} - \frac{z_c}{\sqrt{N-3}}),\ \tanh(z_{ij} + \frac{z_c}{\sqrt{N-3}})] \tag{16}$$

In this equation, N ist he number of samples used for the estimation of ρ_{ij}. The critical value z_c is computed by using the Bonferroni-corrected value for the significance level $\alpha' = \alpha/M$ with M being the number of correlation coefficients to be estimated (see above). The transformed variable z is computed from

$$z_{ij} = \frac{1}{2} \log \frac{1+\rho_{ij}}{1-\rho_{ij}} \tag{17}$$

and the critical value z_c is given by

$$z_c = \Phi^{-1}(1-\alpha'/2) \tag{18}$$

where $\Phi^{-1}(.)$ is the inverse cumulative Gaussian distribution function.

In order to study the effect of LHS on the reduction of statistical uncertainty, a numerical study performing a comparison of the estimation errors (standard deviations) of the correlation coefficients has been carried out. The following table shows confidence interval for a confidence level of 95% as a function of the correlation coefficient ρ and the number of samples N used for one estimation. The statistical analysis is repeated 1000 times. In summary, it turns out that the net effect of LHS compared to PMC is an effective reduction of the sample size by a factor of about 12. For example, as seen from Table 1, it is possible to estimate a coefficient of correlation of $\rho = 0.5$ using 100 samples of LHS with a 95%-confidence interval of 0.101.

Table 1: 95% confidence interval of correlation coefficient, Latin Hypercube Sampling

	ρ				
N	0	0.3	0.5	0.7	0.9
10	0.420	0.382	0.260	0.158	0.035
30	0.197	0.194	0.139	0.073	0.018
100	0.111	0.101	0.071	0.042	0.009
300	0.065	0.057	0.042	0.024	0.006
1000	0.038	0.033	0.025	0.014	0.003

4 Ranking of input variables

For problems involving a large number of input variables it is necessary to reduce the dimension of the random variable space in order to improve the statistical significance of the results. As one possibility it has been suggested to eliminate less important variables based on the adjusted coefficient of determination (COD). This approach utilized an underlying regression model (such as e.g. linear, quadratic w/o mixed terms, or full quadratic) to build an approximation to the experimental input-output relation. The COD can be utilized to indicate the quality and relevance of the approximation by the regression model. The importance of one variable is then measured by the decrease of COD when removing this variable from the regression model. IN order to maintain statistical significance of the results, a rather large value of the COD for the full model is required (> 0.80 is quite recommendable).

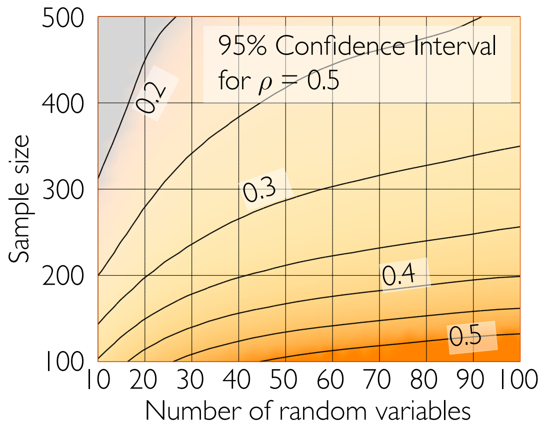


Figure 7: Confidence intervals for coefficients of correlation

The COD R^2 can be conveniently defined n terms of the correlation between the model prediction Z and the actual output data Y:

$$R^2 = \left(\frac{\mathbf{E}[Y \cdot Z]}{\sigma_Y \sigma_Z}\right)^2 \tag{19}$$

Here Z is the regression model, e.g. linear regression in terms of repression coefficients p_i and basis functions $g_i(X)$:

$$Z = \sum_{i=1}^{n} p_i g_i(X) \tag{20}$$

Since the COD approaches 1 if the number of data points equals the number of coefficients, it is useful to adjust the COD by taking into account small sample sizes m. This leads to an R^2_{adj} value:

$$R^2_{adj} = R^2 - \frac{m-1}{m-n}\left(1 - R^2\right) \tag{21}$$

5 Probability-based analysis

5.1 Definition

Generally, failure (i.e. an undesired or unsafe state of the structure) is defined in terms of a limit state function $g(.)$ defining the set $\mathcal{F} = \{\mathbf{X} : g(\mathbf{X})0\}$. Frequently, $Z = g(\mathbf{X})$ is called *safety margin*. The failure probability is defined as the probability of the occurrence of $\mathcal{F}$:

$$P(\mathcal{F}) = P[\{\mathbf{X} : g(\mathbf{X}) \leq 0\}] \tag{22}$$

5.2 FORM - First Order Reliability Method

The FORM-Concept is based on a description of the reliability problem in standard Gaussian space [5]. Hence transformations from correlated non-Gaussian variables $\mathbf{X}$ to uncorrelated

Gaussian variables $\mathbf{U}$ with zero mean and unit variance are required. This concept is especially useful in conjunction with the Nataf-model for the joint pdf of $\mathbf{X}$ [4]. Eventually, this leads to a representation of the limit state function $g(.)$ in terms of the standardized Gaussian variables U_i:

$$g(\mathbf{X}) = g(X_1, X_2, X_n) = g[X_1(U_1, U_n)X_n(U_1, U_n)] \tag{23}$$

This function is linearized with respect to the components in the expansion point $\mathbf{u}^*$. This point is chosen to minimize the distance from the origin in Gaussian space. From this geometrical interpretation it becomes quite clear that the calculation of the design point can be reduced to an optimization problem:

$$\mathbf{u}^* : \mathbf{u}^T\mathbf{u} \rightarrow \text{Min.}; \quad \text{subject to: } g[\mathbf{x}(\mathbf{u})] = 0 \tag{24}$$

Standard optimization procedures can be utilized to solve for the location of $\mathbf{u}^*$ [6]. In the next step, the exact limit state function $g(\mathbf{u})$ is replaced by a linear approximation $\bar{g}(\mathbf{u})$ as shown in Fig. 8. From this, the probability of failure is easily determined to be

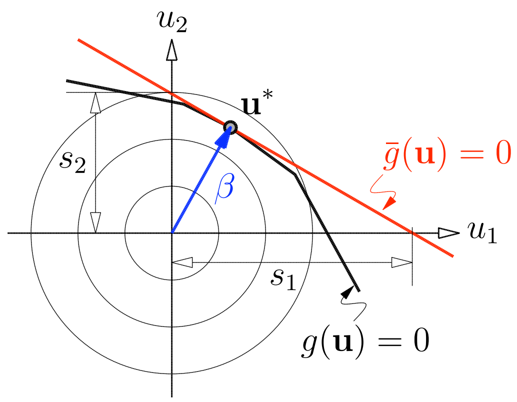


Figure 8: Linearization required for first order reliability method

$$P(\mathcal{F}) = \Phi(-\beta) \tag{25}$$

6 Application example

As an example, consider a simple beam under dynamic loading (cf. Fig. 9). For this beam with length $L = 1$m and a rectangular cross section (w, h) subjected to vertical harmonic loading $F_V(t) = A_V \sin \omega_V t$ and horizontal harmonic loading $F_H(t) = A_H \sin \omega_H t$ the mass should be minimized considering the constraints that the center deflection due to the loading should be smaller than 10 mm. Larger deflections are considered to be serviceability failures. The design variables are bounded in the range $0 < w, h < 0.1$ m. Force values are $A_V = A_H = 300$ N, $\omega_V = 0.20$ rad/s, $\omega_H = 0.15$ rad/s

Using a modal representation of the beam response and taking into account the fundamental vertical and horizontal modes only, the stationary response amplitudes u_V and u_H are readily

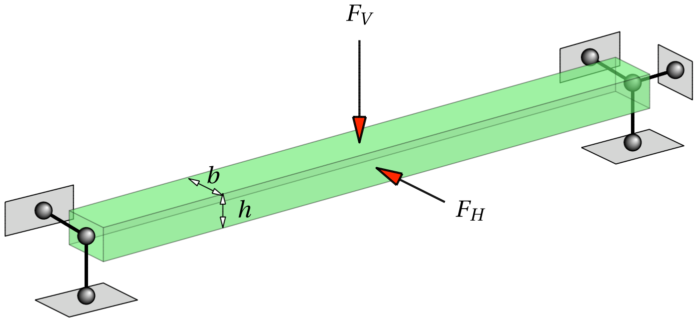


Figure 9: Beam with rectangular cross section

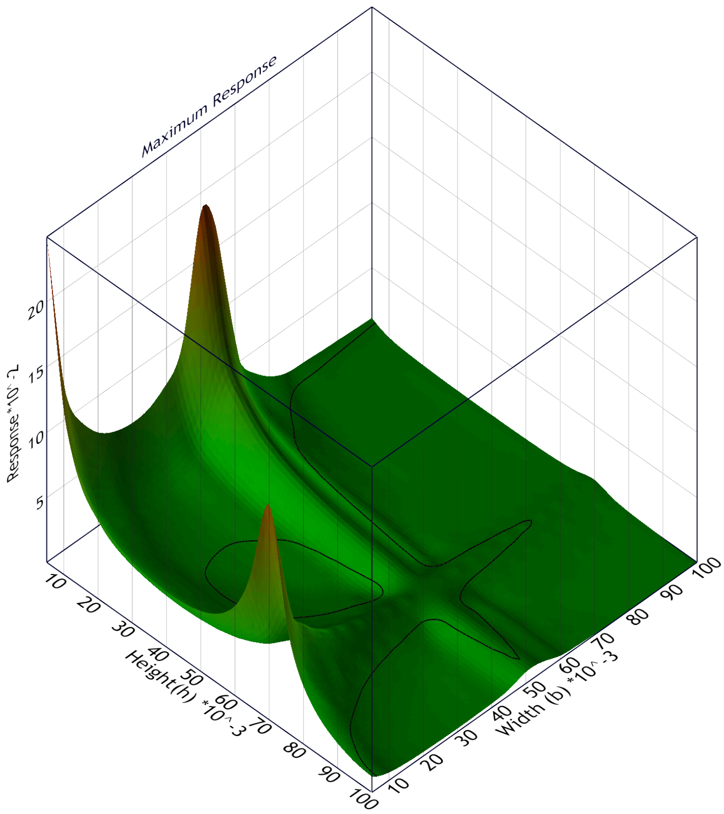


Figure 10: Dynamic response of beam and feasible domain

computed. Fig. 10 shows the maximum of u_V and u_H as a function of the beam geometry. The contour line shown indicates a response value of 0.01 m. Defining this value as acceptable limit of deformation it is seen that the feasible domain is not simply connected. There is an island of feasibility around w = 0.03 m and h= 0.05 m. The deterministic optimum is located on the boundary of this island, i.e. at the values w^* = 0.0138 m and h^* = 0.0483 m.
In the next step, the loading amplitudes are assumed to be log-normally distributed and the excitation frequencies are assumed to be Gaussian random variables. The mean values are assumed to be the nominal values as given above, and the coefficients of variation are assumed to be 10% for the load amplitudes and 5% for the frequencies (Case 1). This implies that the constraints can be satisfied only with a certain probability < 1. Fig. 11 shows the probability $P(\mathcal{F}|w,h)$of violating the constraint as a function of the design variables w and h.

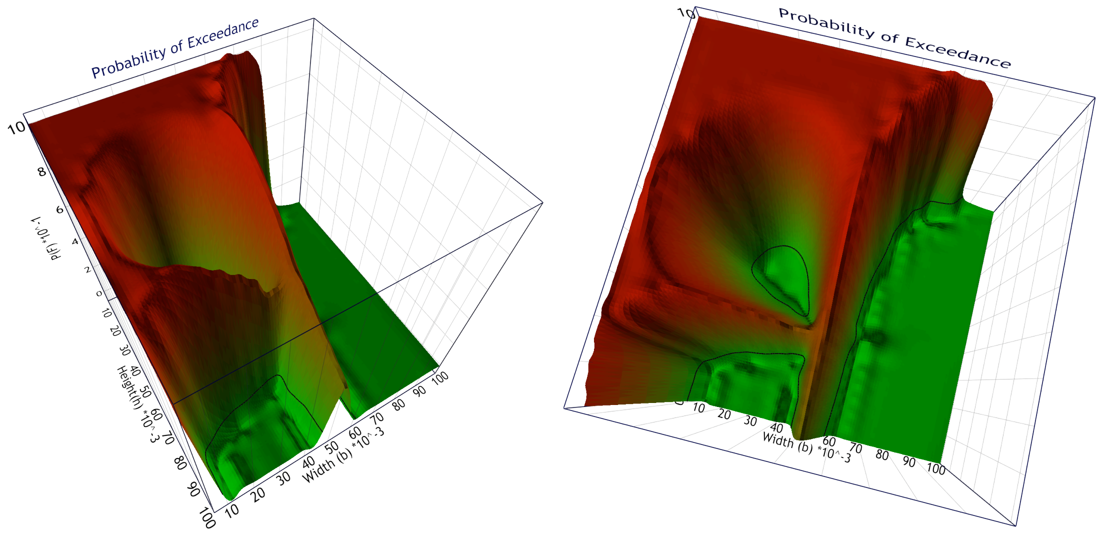


Figure 11: Conditional failure probability $P(\mathcal{F}|w,h)$ depending on w und h, Case 1

Accepting a possible violation of the constraint condition with a probability of 10%, it is seen that the location of the deterministic optimum still contains a probabilistically feasible region. In that sense, the deterministic optimum may be considered as robust.
In a comparative analysis, the coefficients of variation are assumed to be 20% for the load amplitudes and 15% for the frequencies (Case 2). The resulting conditional failure probabilities are shown in Fig. 12. Due to the increased random variability, the feasible region around the deterministic optimum disappeared. This indicates the limited robustness of the deterministic optimum. It is therefore quite important to quantify the uncertainties involved appropriately in order to obtain useful robustness measures.

7 CONCLUDING REMARKS

Structural optimization in the virtual design process tends to lead to highly specialized designs which, unfortunately, frequently lack robustness of performance with respect to inherent uncer-

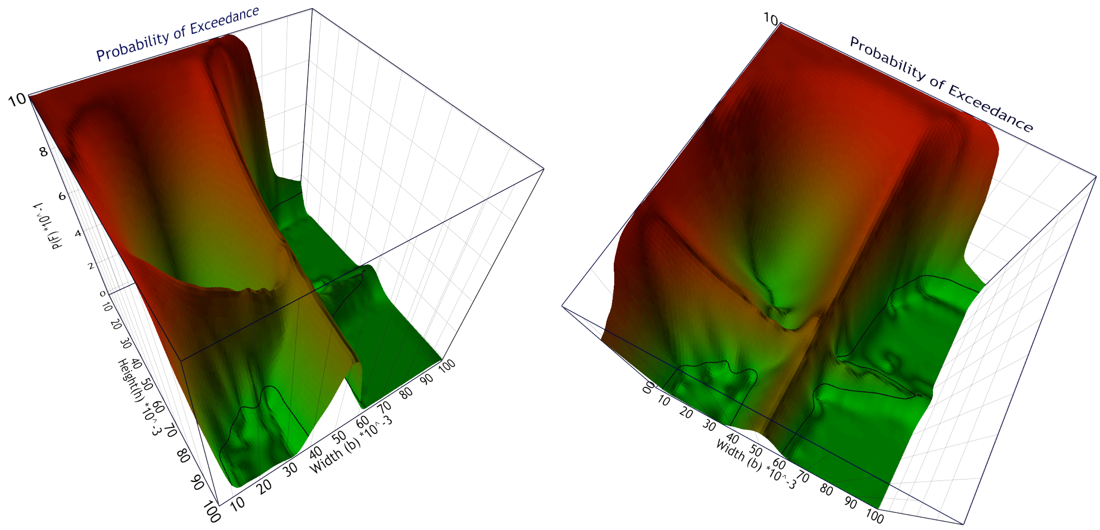


Figure 12: Conditional failure probability $P(\mathcal{F}|w,h)$ depending on w und h, Case 2

tainties. Important reasons for such situations lie in the inherent randomness of either design parameters or constraint conditions. One possible way to overcome this problem is the application of robustness-based optimization. This allows to take into account random variability in the problem formulation thus leading to optimal designs which are automatically robust. It appears that this concept should be applicable to a large number of structural optimization problems. However, the numerical effort to carry out the analysis is quite substantial. Further research into the reduction of effort is therefore required.

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