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Lectures

Adaptive Moving Least Square Approximation for the Design Reliability Analysis

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Adaptive Moving Least Square Approximation for the Design Reliability Analysis

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Abstract

A large number of problems in manufacturing processes, production planning, finance and engineering design require an understanding of potential sources of variations and quantification of the effect of variations on product behavior and performance. Traditionally, in engineering problems uncertainties have been formulated only through coarse safety factors. Such methods often lead to overdesigned products.

Different methods exist to describe model uncertainties and to calculate reliability and safety, but they all have a limited area of application.

A new adaptive response surface method is introduced to analyse the design reliability with high accuracy and efficiency. Whereby the surrogate model is based on an improved moving least square approximation combined with an adaptive design of experiment. In order to obtain a fast simulation procedure on the response surface an adaptive importance sampling concept is used.

Two numerical examples show the applicability of this concept for highly nonlinear state and limit state functions and multiple design points and separated unsafe domains. The probabilistic analysis tasks are performed with the optiSLang software package.

Keywords: reliability analysis, adaptive response surfaces, moving least square approximation, adaptive design of experiment, adaptive importance sampling

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Property	SD/Mean %
Metallic materials, yield	15
Carbon fiber composites, rupture	17
Metallic shells, buckling strength	14
Junction by screws, rivet, welding	8
Bond insert, axial load	12
Honeycomb, tension	16
Honeycomb, shear, compression	10
Honeycomb, face wrinkling	8
Launch vehicle , thrust	5
Transient loads	50
Thermal loads	7.5
Deployment shock	10
Acoustic loads	40
Vibration loads	20

Table 1: Sources of uncertainties (Klein et al. (1994)) given by standard deviation (SD) and mean value as shown in Figure 1.

1 Introduction

Within many engineering fields, structural design must meet challenging demands of cost-effectiveness. This leads to light-weight, highly flexible, and consequently vulnerable structures. In order to assess potential risks associated with such a design, the structural analysis must take into account available information on the randomness of loads and structural properties. It means that the analysis should include reliability estimates in an appropriate manner.

Considering the properties of the computational analysis realistically it is necessary to take into account some uncertainty. This uncertainty can be conveniently described in terms of probability measures, such as distribution functions. Probabilistic analysis typically involves two areas of statistical variability as shown in Table 1. The first group consists of the uncontrollable uncertainties and tolerances. These include material property variability, manufacturing process limitations, environmental variability, such as temperature, operating processes (misuse) and result scatter arising from deterioration. The second group – the controllable parameters – involves design configurations, geometry, loads.

Although Monte Carlo methods are most versatile, intuitively clear, and well understood, the computational cost (the number of computational runs required) in many cases is prohibitive. Thus approximations become important, which can be based e.g. on the response surface methods (RSM) or first/second order reliability methods (FORM/SORM). For the feasibility of response surface approaches it is quite essential to reduce the number of variables to a tractable amount. This may require extensive sensitivity analysis in order to identify the relevant random variables. This is particularly important for random variables arising from discretization of random fields or processes. In this context, close coupling between the tools for stochastic and computational analyses is essential. Simplifications can be based on the following items

- Global variance-based sensitivity or robustness analysis of the structural response

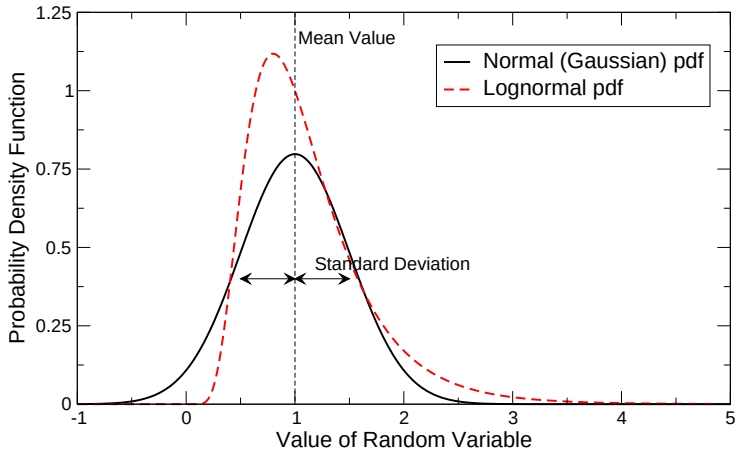


Figure 1: Probability density function $f_X(x)$ of the normal and lognormal distribution with mean and standard deviation.

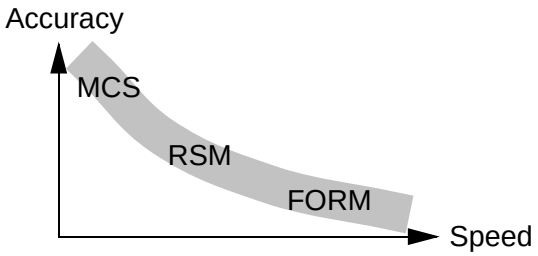
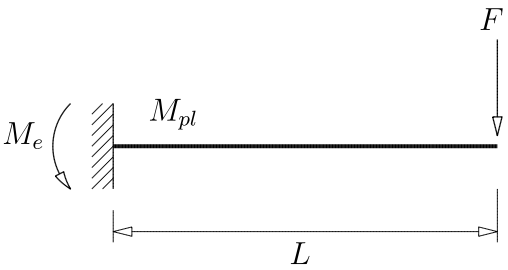


Figure 2: Performance of methods for stochastic analysis. a.) Monte Carlo simulation (MCS), b.) Response Surface Method (RSM), c.) First Order Reliability Method (FORM).



$$\mathcal{F} = \{(F, L, M_{pl}) : FL \geq M_{pl}\} = \{(F, L, M_{pl}) : 1 - \tfrac{FL}{M_{pl}} \leq 0\}$$

Figure 3: Structural system and several unique failure conditions.

with respect to the random variables. Again, this aims at reducing the number of random variables needed.

- Concentrate random sampling in the region which contributes most to the total failure probability. This is generally called "importance sampling". It is important to note that most importance sampling strategies work best with a low number of random variables.

- Approximation of the numerical response by a class of simple mathematical functions. This is the so-called "response surface method". Again, it is vital that the number of random variables be kept small.

As a very simplistic rule-of-the-thumb, Fig. 2 gives the accuracy/speed ratio for some solution methods as mentioned above. However, there are situations in which MCS can be comparably fast or FORM can be comparably accurate.

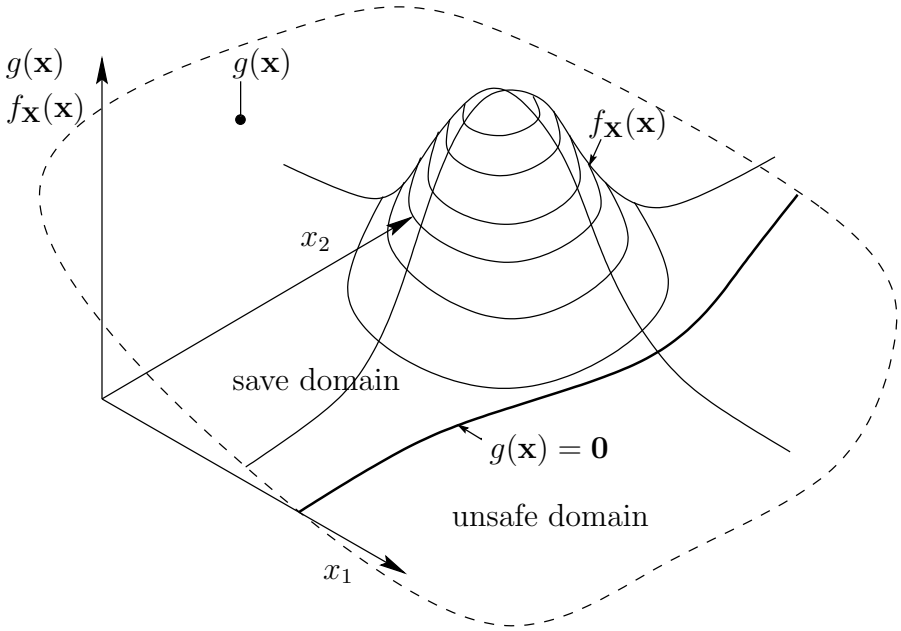


Figure 4: The state function $g(\mathbf{x})$ of a numerical model is given implicitly, e.g. is result of a finite element analysis depending on several design responses. The failure condition leads to a unknown deterministic limit state function $g(\mathbf{x}) = 0$, where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function.

2 Reliability Analysis

Safety and reliability analysis warrants the exclusion of damage and design collapse during the life time. Probability of surviving is the numerical quantity of safety and reliability and the probability of failure is the complement.

We can define any undesired or unsafe state of a response as an event $\mathcal{F}$ out of the set of all random variables $\mathbf{X}$ such a way that the assign state function $g(\mathbf{x})$ is less or equal to zero. Generally, failure (i.e. an undesired or unsafe state of the response) is defined in terms of a limit state function $g(.)$, i.e. by the set $\mathcal{F} = \{\mathbf{X} : g(\mathbf{X}) \leq 0\}$. Frequently, $Z = g(\mathbf{X})$ is called *safety margin*.

As indicated in Fig. 3, the definition of the limit state function is not unique. The failure probability is defined as the probability of the occurrence of the unsafe event $\mathcal{F}$:

$$P(\mathcal{F}) = P[\{\mathbf{X} : g(\mathbf{X}) \leq 0\}] \tag{1}$$

This quantity is *unique*, i.e. not depending on the particular choice of the limit state function. The response behavior near the failure state is most important in the reliability analysis. The random design parameters, such as loadings, material parameters and geometry, are the set of basic random variables $\mathbf{X}$ which determine the probabilistic response of numerical model. The failure condition is defined by a deterministic limit state function

$$g(\mathbf{x}) = g(x_1, x_2, \ldots, x_n) \leq 0$$

as shown in Fig. 2. The failure probability of a design is given by

$$P(\mathcal{F}) = P[\mathbf{X} : g(\mathbf{X}) \leq 0] = \int_{g(\mathbf{x})\leq 0} \overset{n}{\ldots} \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{2}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of the basic random variables.

3 Response Surface Methods

3.1 Global Polynomial Approximation

Although Monte Carlo methods are most versatile, intuitively clear, and well understood, the computational cost (the number of finite element runs required) is in many cases prohibitive. Thus approximations become important which can be based e.g. on the response surface method. Normally, the state function $g(\mathbf{X})$ of a system response is described implicitly, e.g. through an algorithmic procedure within finite element analysis.

Alternatively, the original state function can be approximated by a response surface function $\tilde{g}(\mathbf{x})$ which has polynomial form (Rackwitz (1982); Faravelli (1986); Bucher and Bourgund (1987, 1990); Engelund and Rackwitz (1992); Rajashekhar and Ellingwod (1993)).

A commonly used method for response value approximation is the regression analysis. Usually, the approximation function is a first order or second order polynomial (Box and Draper (1987); Myers (1971)). As an example in the $(n = 2)$-dimensional case, k-responses $(k = 1, ..., m)$ will be approximated using a least square quadratic polynomial in the following form:

$$\tilde{g}_k(\mathbf{x}) = \beta_1 x_{1k} + \beta_2 x_{2k} + \beta_{11} x_{1k}^2 + \beta_{22} x_{2k}^2 + 2\beta_{12} x_{1k} x_{2k} + \epsilon_k \tag{3}$$

Herein the term ϵ_k represents the approximation errors. The approximate coefficients β can be calculated using the least square postulate

$$S = \sum_{k=1}^{m} \epsilon_k^2 = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} \rightarrow \min$$

Additional, the limit state function $g(\mathbf{x}) = 0$ themselves can be interpolated by second order polynomials (Ouypornprasert and Bucher (1988); Bucher et al. (1988)).

One of the major advantages of the response surface method lies in its potential to selectively determine the number of structural analyses of the support points. This is especially helpful if some overall knowledge on the system behavior - particularly near to the failure region - is a priori available. By such means the computational effort can be substantially reduced.

On the other hand, the global approximation schemes widely used in the application of the response surface method can be quite misleading due to the lack of information in certain regions of the random variable space. Standard second order polynomial approximations are not sufficiently flexible. So, the estimation of the failure probability using this global approximation leads to large errors, in particular for small failure probabilities $P(\mathcal{F}) < 10^{-2}$ and a number of random parameters of $n > 5$. It is therefore required to avoid such undesirable approximation errors at reasonable computational effort.

3.2 Adaptive Response Surface Method

3.2.1 Moving least square approximation

A commonly used approximation method with minimized the regression error within the support point values is the moving least square method. The main advantage of this method is the flexibility for the approximation of highly nonlinear state and limit state functions. The proposed method is suitable for computing the reliability of complex models and is intended to provide reasonably accurate estimates of failure probabilities while maintaining computational efficiency.

Moving least square (MLS) functions can approximate locally clustered support point samples with higher local approximation quality. In addition MLS improve the response surface model using additional support points. MLS is formulated as

$$\hat{y}(\mathbf{x}) = \sum_{i=1}^{n_b} h_i(\mathbf{x}) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x})\ \mathbf{a}(\mathbf{x}) \tag{4}$$

with a predefined number of basis terms n_b, a vector of basis functions $\mathbf{h}$ and the associated vector of the coefficients $\mathbf{a}$. Lancaster and Salkauskas (1986) formulates a local MLS approximation as

$$\hat{y}(\mathbf{x}, \mathbf{x}_j) = \sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}_j)\ \mathbf{a}(\mathbf{x})$$

with $j = 1, ..., n_s$ support points. The approximate coefficient vector $\mathbf{a}$ can be calculated using the weighted least square postulate

$$\begin{aligned} S(\mathbf{x}) &= \sum_{j=1}^{n_s} w(\mathbf{x} - \mathbf{x}_j) \left(\hat{y}(\mathbf{x}, \mathbf{x}_j) - y(\mathbf{x}_j)\right)^2 \\ &= \sum_{j=1}^{n_s} w(\mathbf{x} - \mathbf{x}_j) \left(\sum_{i=1}^{n_b} h_i(\mathbf{x}_j) a_i(\mathbf{x}) - y(\mathbf{x}_j) \right)^2 \\ &= (\mathbf{H}\mathbf{a} - \mathbf{g})^T \mathbf{W}(\mathbf{x}) (\mathbf{H}\mathbf{a} - \mathbf{g}) \rightarrow \min \end{aligned} \tag{5}$$

with the weighting function $w(\mathbf{x} - \mathbf{x}_j)$ and

$$\begin{aligned} \mathbf{g} &= [y(\mathbf{x}_1) \quad y(\mathbf{x}_2) \quad ... \quad y(\mathbf{x}_{n_s})]^T \\ \mathbf{H} &= [\mathbf{h}(\mathbf{x}_1) \quad \mathbf{h}(\mathbf{x}_2) \quad ... \quad \mathbf{h}(\mathbf{x}_{n_s})]^T \\ \mathbf{h}(\mathbf{x}_j) &= [h_1(\mathbf{x}_j) \quad h_2(\mathbf{x}_j) \quad ... \quad h_{n_b}(\mathbf{x}_j)]^T \\ \mathbf{W}(\mathbf{x}) &= \mathrm{diag}[w(\mathbf{x} - \mathbf{x}_1) \quad w(\mathbf{x} - \mathbf{x}_2) \quad ... \quad w(\mathbf{x} - \mathbf{x}_{n_s})] \end{aligned}$$

The least square error $S(\mathbf{x})$ may be a minimum in case that the partial gradients are zero.

$$\frac{\partial S(\mathbf{x})}{\partial \mathbf{a}} = 0$$

So using the equation (5) a linear equation system gives an estimation of the coefficient vector $\mathbf{a}$

$$\mathbf{a}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x})\ \mathbf{B}(\mathbf{x})\ \mathbf{g} \tag{6}$$

with

$$\mathbf{M}(\mathbf{x}) = \mathbf{H}^T\ \mathbf{W}(\mathbf{x})\ \mathbf{H}$$
$$\mathbf{B}(\mathbf{x}) = \mathbf{H}^T\ \mathbf{W}(\mathbf{x})$$

Cause the matrix of the basis function $\mathbf{M}(\mathbf{x})$ should be non-singular always a sufficient number of n_s immediate neighbor support points have to be available. The number must be at least as large as number of the basis terms. The equation (6) inserted in (4) gives the approximation function

$$\hat{y}(\mathbf{x}) = \mathbf{h}^T(\mathbf{x})\ \mathbf{M}^{-1}(\mathbf{x})\ \mathbf{B}(\mathbf{x})\ \mathbf{g}$$

An accurate as possible approximation quality requires a weighting function which is larger than zero $w(\mathbf{x} - \mathbf{x}_j) > 0$ and monotonically decreasing $w(\|\mathbf{x} - \mathbf{x}_j\|)$ inside of a small sub space $\Omega_s \subset \Omega$. So the influence of supports far from the actual coordinates is unimportant. A uniform weighting is given by a symmetry condition $w(\mathbf{x} - \mathbf{x}_j) = w(\mathbf{x}_j - \mathbf{x}) = w(\|\mathbf{x} - \mathbf{x}_j\|)$. Usually, an exponential function is used in this way:

$$w(\|\mathbf{x} - \mathbf{x}_j\|) = \begin{cases} e^{-\left(\frac{\|\mathbf{x} - \mathbf{x}_j\|}{D\alpha}\right)^2} & \|\mathbf{x} - \mathbf{x}_j\| \leq D \\ 0 & \|\mathbf{x} - \mathbf{x}_j\| > D \end{cases} \tag{7}$$

with a constant

$$\alpha = \frac{1}{\sqrt{-\ log\ 0.001}}$$

and a influence radius D to choose. It is obvious that the smaller D the better the response values of the support points fit the given values. But as mentioned above at least n_b support points have to be available in every point to be approximated. Therefore it is possible that a D has to be chosen which leads to a large shape function error at the support points.

To avoid these problems a new regularized weighting function was introduced by Most and Bucher (2005):

$$w_R(\|\mathbf{x} - \mathbf{x}_j\|) = \begin{cases} \dfrac{\hat{w}_R(\|\mathbf{x} - \mathbf{x}_j\|)}{\sum_{i=1}^{n_s} \hat{w}_R(\|\mathbf{x} - \mathbf{x}_i\|)} & \|\mathbf{x} - \mathbf{x}_j\| \leq D \\ 0 & \|\mathbf{x} - \mathbf{x}_j\| > D \end{cases} \tag{8}$$

with

$$\hat{w}_R(d) = \frac{\left(\left(\frac{d}{D}\right)^2 + \varepsilon\right)^{-2} - (1+\varepsilon)^{-2}}{(\varepsilon)^{-2} - (1+\varepsilon)^{-2}}\ ;\ \varepsilon \ll 1 \tag{9}$$

It is recommended by the authors to use the value

$$\varepsilon = 10^{-5}$$

This new regularized weighting function works better than the exponential function. But if the ratio of the minimal distance among the supports to the extent of areas where are

Table 2: Numbers of required support points for different DOE schemes

	Number of support points[a]						
	Linear approximation			Quadratic approximation[b]			
Number of Variables n	Koshal Linear	D–optimal[c] (linear)	Full factorial ($m = 2$)	Koshal Quadr.	D–optimal[d] (quadr.)	Full factorial ($m = 3$)	Central composite (CCD)
1	2	2	2	3	3	3	3
2	3	4	4	6	9	9	9
3	4	6	8	10	15	27	15
4	5	8	16	15	23	81	25
5	6	9	32	21	32	243	43
6	7	11	64	28	42	729	77
7	8	12	128	36	54	2187	143
8	9	14	256	45	68	6561	273
9	10	15	512	55	83	19683	531
10	11	17	1024	66	99	59049	1045
11	12	18	2048	78	117	177147	2071
12	13	20	4096	91	137	531441	4121
13	14	21	8192	105	158	1594323	8219
14	15	23	16384	120	180	4782969	16413
15	16	24	32768	136	204	14348907	32799

[a]Including only one center point ($n_c = 1$).
[b]Also usable with linear approximation approach.
[c]Based on full factorial DOE ($m = 2$), with 1.5 times linear koshal designs.
[d]Based on full factorial DOE ($m = 3$), with 1.5 times quadratic koshal designs.

no supports becomes worse the same problems occur again. As a matter of fact a large D is needed to approximate for coordinates where no support points are around and a small D is needed for coordinates where are a lot of support points in order to reach a minimal approximation error. To comply with this conditions it is necessary to use a function $d(\mathbf{x})$ for the influence radius instead of a constant D.

3.2.2 Adaptive design of experiment

In particular, these response surfaces can be adaptively refined to consistently increase the accuracy of the estimated failure probability. This is especially suitable for the reliability analysis of complex nonlinear structures. An arbitrary number of check points even in high local concentration can be used without approximation problems. Using deterministic design of experiment, the necessary number of support points become very high with an increasing number of random variables, as shown in Table 2.

To decrease the number of support points in an optimized way, the so called D-optimality criterion is used. A discussion of this criterion is presented by Box and Draper (1971). The effectiveness of a design in satisfying the minimum variance (D-Optimal) criterion is expressed by the D-Efficiency of the design. In Myers and Montgomery (1995), more exact specifications of the D-Optimal criteria and further criteria called alphabetic optimality criteria are described. However, the first design of experiment in the first

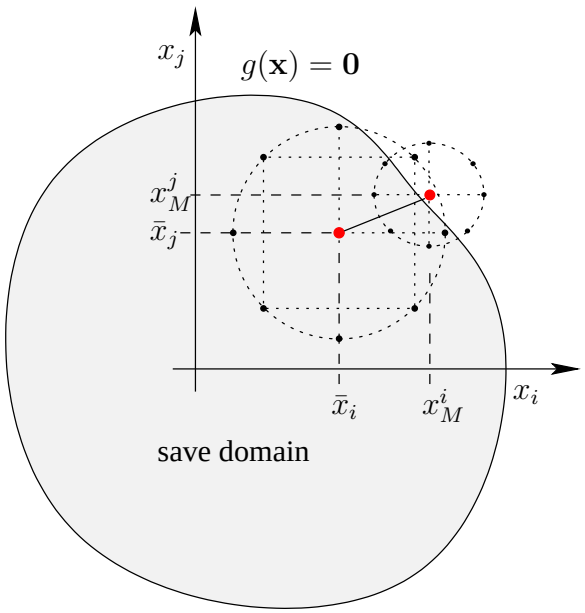


Figure 5: Adaptive design of experiment in the random space.

iteration should explore the random space including safe and unsafe domain as accurate as possible. A possible approach is given in Klingmüller and Bourgund (1992) with

$$x_i = \bar{x}_i \pm f\sigma_{x_i}$$

whereby

$$f = \Phi^{-1}(P(\mathcal{F})) = 3, \ldots, 5$$

is a factor depending on the assumed failure probability. Bucher and Bourgund (1987) give an efficient possibility to adaption a design of experiment in the next iterations with

$$\mathbf{x}_M = \bar{\mathbf{x}} + (\mathbf{x}_D - \bar{\mathbf{x}})\frac{g(\bar{\mathbf{x}})}{g(\bar{\mathbf{x}}) - g(\mathbf{x}_D)}$$

with

$$\mathbf{x}_D = E[\mathbf{X}|g(\mathbf{x}) \leq 0]$$

as shown in Figure 3.2.1. This is achieved by a combination of random search strategies (based on the adaptive sampling approach, see Bucher (1988) for details) as well as deterministic search refinement. In such a way for the most practical examples 3 till 6 iteration steps are necessary for a sufficient convergence. So this adaptive design of experiment using a D-optimal linear or quadratic design in combination with the improved moving least square approximation is suitable up to $n \leq 20$ random parameters.

4 Examples

4.1 The Quattro Function

Within the first example the different results of the presented adaptive response surface method in comparison with a classic global response surface method are given. The state

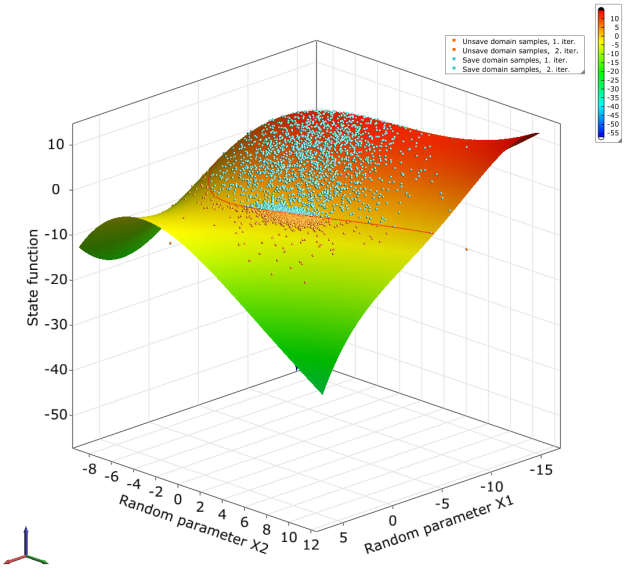


Figure 6: The Quattro Function. Analytical state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$ in combination with direct importance adaptive sampling.

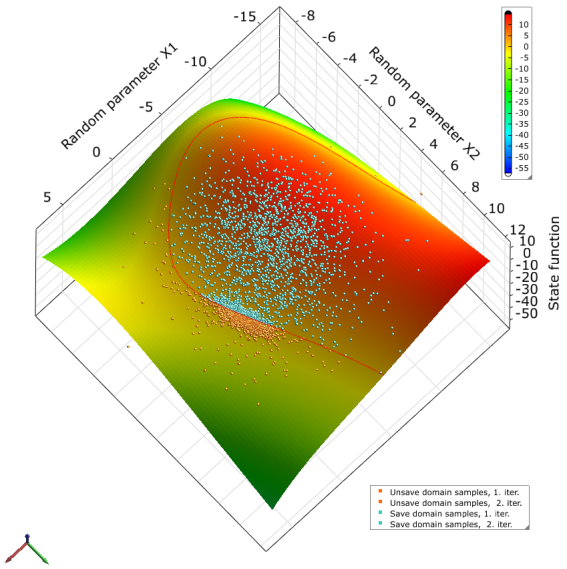


Figure 7: The limit state function $g(\mathbf{x}) = 0$ is a high-curved nonlinear one. The random parameters X_1 and X_2 are normal distributed variables with mean $\bar{X}_1 = -3.9$, and $\bar{X}_2 = 1.5$ and standard deviation $\sigma_{X_1} = \sigma_{X_2} = 1$.

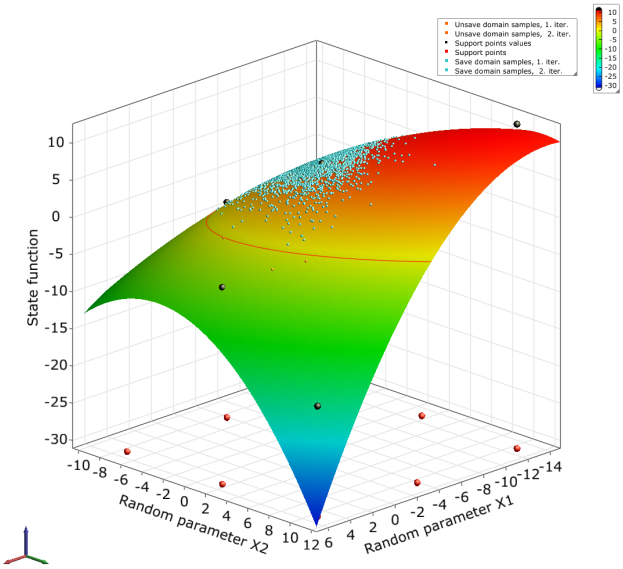


Figure 8: Approximated state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$ using the global 2nd order polynomial approximation using a global central composite design of experiment.

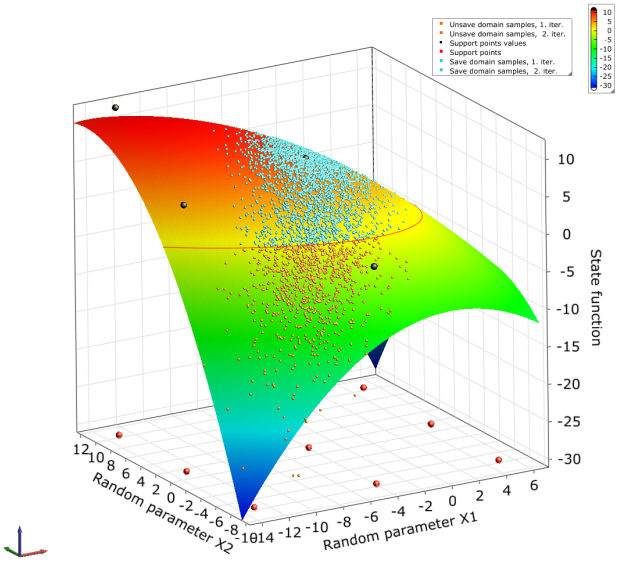


Figure 9: Using the global 2nd order polynomial approximation a wrong most probability failure domain is identified using importance adaptive sampling.

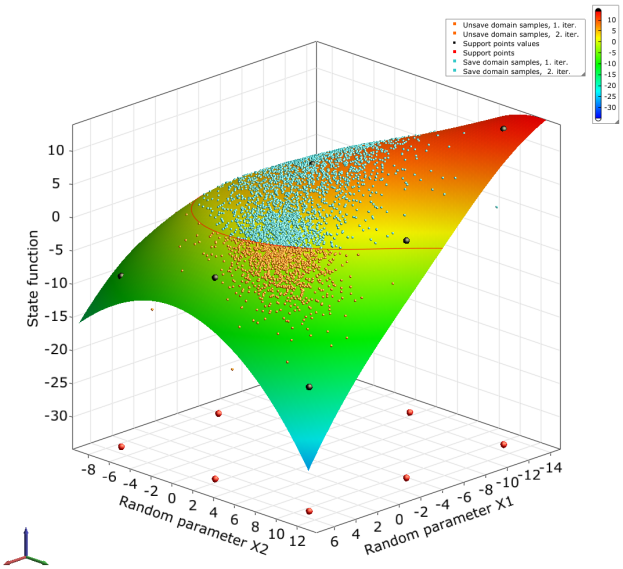


Figure 10: Approximated state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$ using the adaptive response surfaces with importance adaptive sampling and the first design of experiment (central composite design).

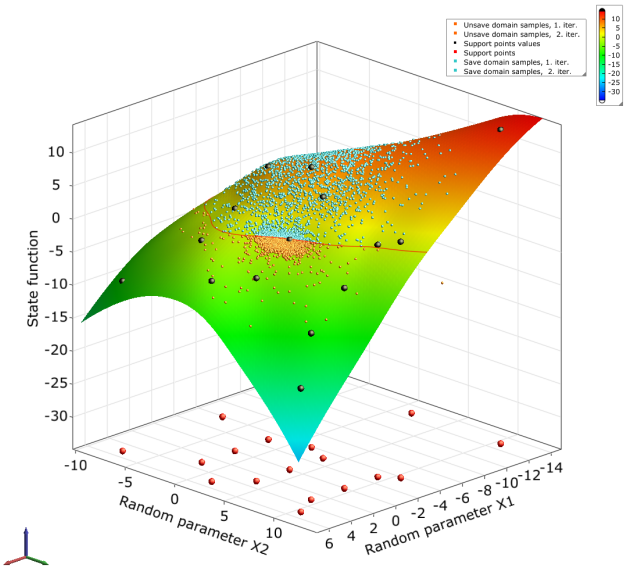


Figure 11: Adaptive moving least square approximation with first and second design of experiment.

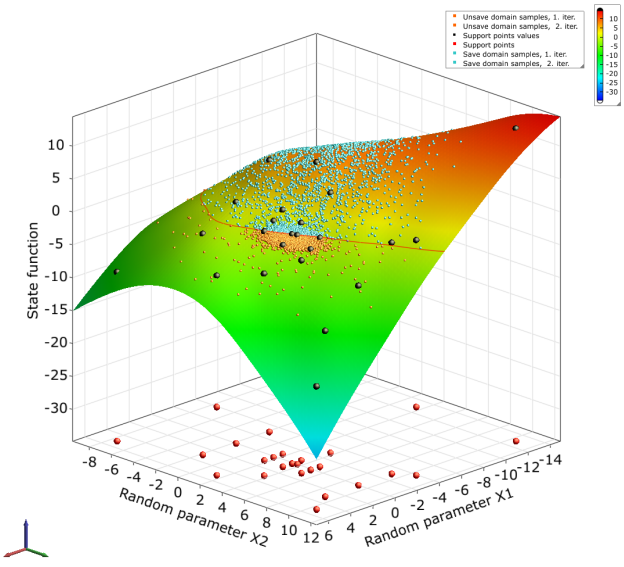


Figure 12: Adaptive moving least square approximation with first till third design of experiment.

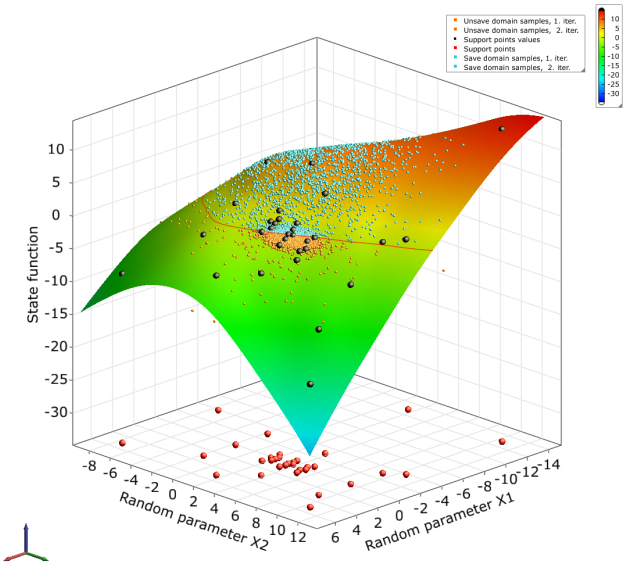


Figure 13: Adaptive moving least square approximation with first till fourth design of experiment.

	Number of state function evaluations	Failure probability	Variance of the estimation $\sigma^2_{\bar{P}(\mathcal{F})}$	Accuracy error %
Direct adaptive importance sampling	4000	$4.8907 \cdot 10^{-6}$	$1.5813 \cdot 10^{-7}$	0
Global polynomial 2nd order approximation	9	$2.1563 \cdot 10^{-10}$	$2.5303 \cdot 10^{-11}$	2268000
Adaptive response surface approximation	9	$7.9728 \cdot 10^{-11}$	$6.0763 \cdot 10^{-12}$	6134100
	18	$4.47 \cdot 10^{-6}$	$1.1713 \cdot 10^{-7}$	9
	27	$4.4096 \cdot 10^{-6}$	$1.214 \cdot 10^{-7}$	11
	36	$4.6044 \cdot 10^{-6}$	$1.1337 \cdot 10^{-7}$	6

Table 3: Results of the failure probability of the Quattro Function depending on the applied analysis method.

function

$$g(x_1, x_2) = -\left(\frac{x_1}{6} + \frac{1}{2}\right)^4 - \left(\frac{x_2}{8} - \frac{6}{5}\right)^4 - \left(\frac{2}{9}x_1 + \frac{1}{2}\right)^3 \left(\frac{x_2}{8} - \frac{6}{5}\right) + \left(\frac{x_1}{6} + \frac{1}{2}\right)^2 + \left(\frac{x_2}{8} - \frac{6}{5}\right)^2 - \frac{5}{3}x_1 - \frac{3}{4}x_2 + \frac{11}{5}$$

is a two-dimensional fourth order polynomial, as shown in Figure 6. Furthermore, the limit state function $g(\mathbf{x}) = 0$ is a high-curved nonlinear one, as shown in Figure 7. The random parameters X_1 and X_2 are normal distributed variables with mean $\bar{X}_1 = -3.9$, and $\bar{X}_2 = 1.5$ and standard deviation $\sigma_{X_1} = \sigma_{X_2} = 1$. In order to obtain the reference failure probability a direct importance adaptive sampling is done with $N = 2000$ samples, two iterations and an initial variance multiplier $\sigma^{(i=1)} = 3$. The given failure probability is $P(\mathcal{F}) = 4.8907 \cdot 10^{-6}$ with a standard error of $\sigma_{\bar{P}(\mathcal{F})} = 1.5813 \cdot 10^{-7}$. In addition, the same adaptive sampling procedure is used to calculate the failure probability on the response surfaces.

For this example, the approximation using the global 2nd order polynomial approximation and a global central composite design of experiment leads to an error of 2268000 % in calculation the failure probability, as shown in the Figures 8 and 9. Using the global 2nd order polynomial approximation a wrong most probability failure domain is identified using importance adaptive sampling.

Applying the new adaptive response surface method to this state function, as shown in Figures 10 till 13, leads to accurate estimation of the failure probability already after the first adaption with $N = 18$ state function evaluations. In summary, using three adaptions of the central composite design with in total $N = 36$ state function evaluations the error of the given failure probability is 6% only (see Table 3 for details).

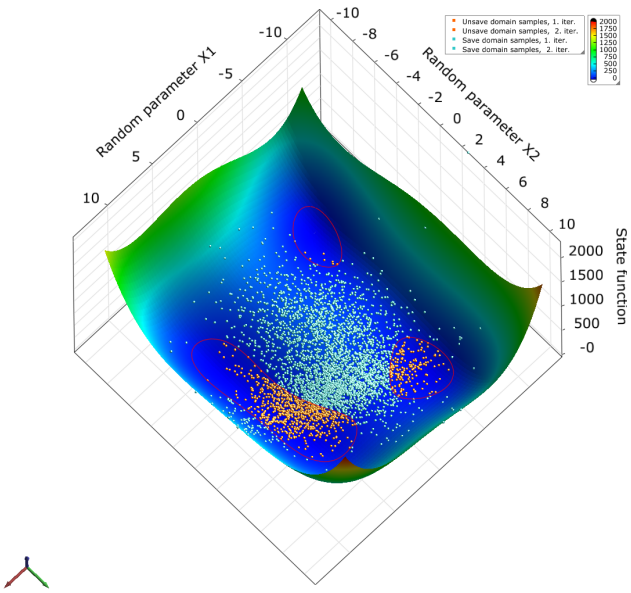


Figure 14: The Himmelblau Function. Analytical state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$ with direct importance adaptive sampling. The random parameters X_1 and X_2 are normal distributed variables with mean $\bar{X}_1 = -0.6$, and $\bar{X}_2 = 0$ and standard deviation $\sigma_{X_1} = \sigma_{X_2} = 1$.

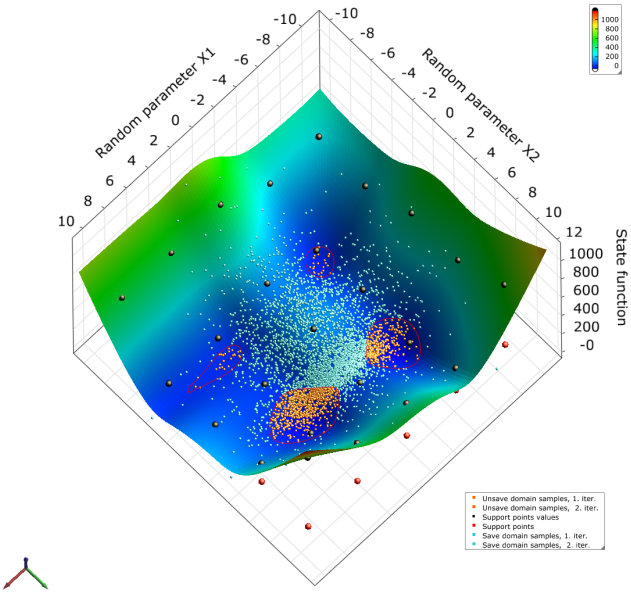


Figure 15: Approximated state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$ using the adaptive response surfaces with importance adaptive sampling and the first design of experiment (full factorial design).

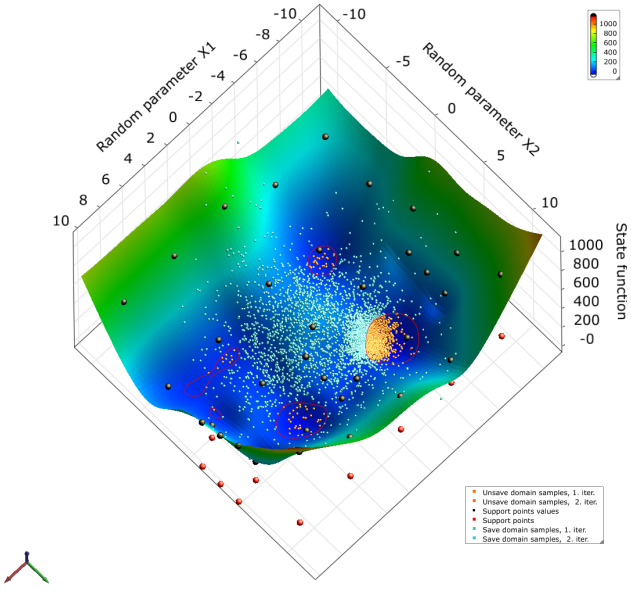


Figure 16: Adaptive moving least square approximation with first and second (central composite) design of experiment.

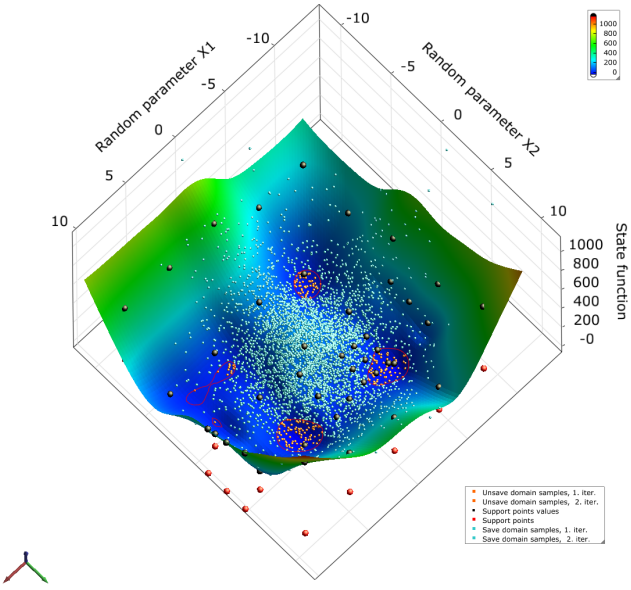


Figure 17: Adaptive moving least square approximation with first till third (central composite) design of experiment.

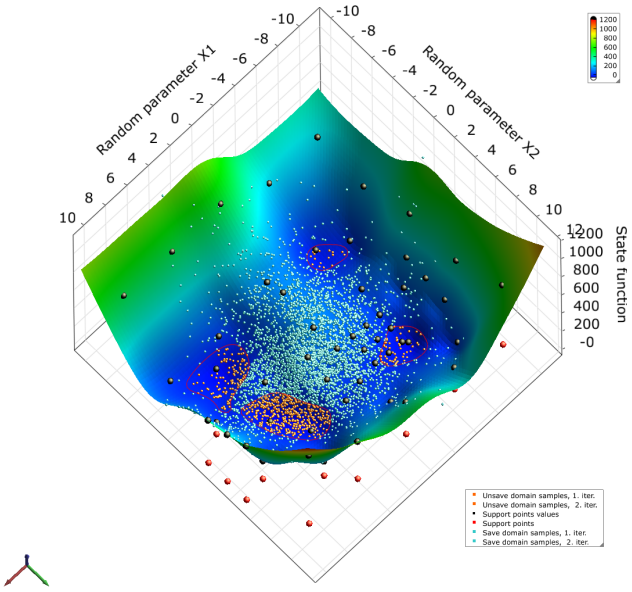


Figure 18: Adaptive moving least square approximation with first till fourth (central composite) design of experiment.

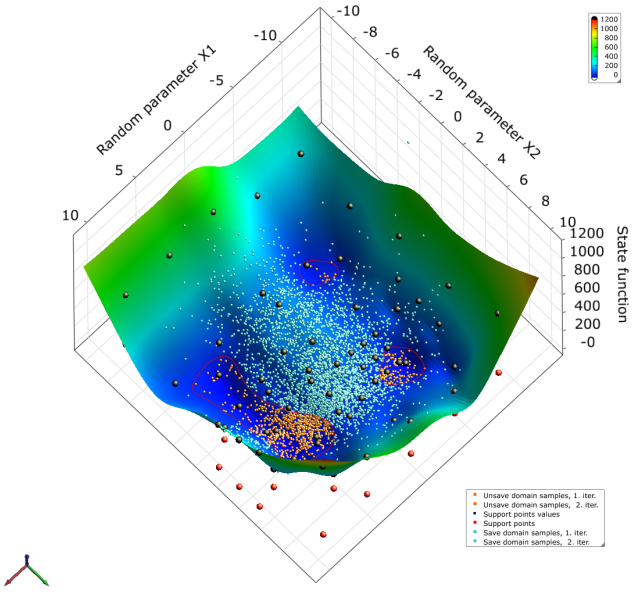


Figure 19: Adaptive moving least square approximation with first till fifth (central composite) design of experiment.

	Number of state function evaluations	Failure probability	Variance of the estimation $\sigma^2_{\hat{P}(\mathcal{F})}$	Accuracy error %
Direct adaptive importance sampling	4000	$3.6817 \cdot 10^{-6}$	$3.5149 \cdot 10^{-7}$	0
Adaptive response surface approximation	25	$4.5943 \cdot 10^{-4}$	$2.6486 \cdot 10^{-5}$	99
	34	$1.05 \cdot 10^{-4}$	$4.2485 \cdot 10^{-6}$	96
	43	$3.3566 \cdot 10^{-6}$	$7.2673 \cdot 10^{-7}$	10
	52	$7.9314 \cdot 10^{-6}$	$1.0342 \cdot 10^{-6}$	54
	61	$3.7946 \cdot 10^{-6}$	$3.3768 \cdot 10^{-7}$	3

Table 4: Results of the failure probability of the Himmelblau Function depending on the applied analysis method.

4.2 The Himmelblau Function

A commonly used fourth order polynomial test function within the nonlinear optimization is the so-called Himmelblau (1972) function

$$g(x_1, x_2) = \left(\frac{x_1^2}{1.81} + \frac{x_2}{1.81} - 11 \right)^2 + \left(\frac{x_1}{1.81} + \frac{x_2^2}{1.81} - 7 \right)^2 - 50$$

with multiple separated unsafe domains. The Figure 14 shows the Himmelblau state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$. The random parameters X_1 and X_2 are normal distributed variables with mean $\bar{X}_1 = -0.6$, and $\bar{X}_2 = 0$ and standard deviation $\sigma_{X_1} = \sigma_{X_2} = 1$. Using direct importance adaptive sampling the reference failure probability is $P(\mathcal{F}) = 3.6817 \cdot 10^6$ with a standard error of $\sigma_{\bar{P}(\mathcal{F})} = 3.5149 \cdot 10^{-7}$. Contingent on the existence of multiple separated unsafe domains other analysis methods existing in optiSLang like directional sampling (Bjerager (1988); Ditlevsen and Bjerager (1989); Melchers (1990)) or FORM and ISPUD are not applicable.

Applying adaptive response surface method to this state function, as shown in Figures 15 till 19, leads to accurate estimation of the failure probability already after the fourth adaption with in total $N = 61$ state function evaluations. The Figure 15 shows the approximated state function $g(\mathbf{x})$ and limit state function $g(\mathbf{x}) = 0$ using the first design of experiment (full factorial design with $N = 25$ state function evaluations). In summary, using four adaptions of the central composite design with $N = 9$ state function evaluations per iteration the error of the given failure probability is 3% only (see Table 4 for details).

5 Concluding Remarks

A new adaptive response surface method is introduced to analyse the design reliability with high accuracy and efficiency. Whereby the surrogate model is based on an improved moving least square approximation combined with an adaptive design of experiment. In order to obtain a fast simulation procedure on the response surface an adaptive importance sampling concept is used. In this sense, the proposed method is very robust and efficient for every safety level up to $n \leq 20$ random parameters and combine the advantages of an adaptive design of experiment, adaptive sampling and efficient response surface methods.

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