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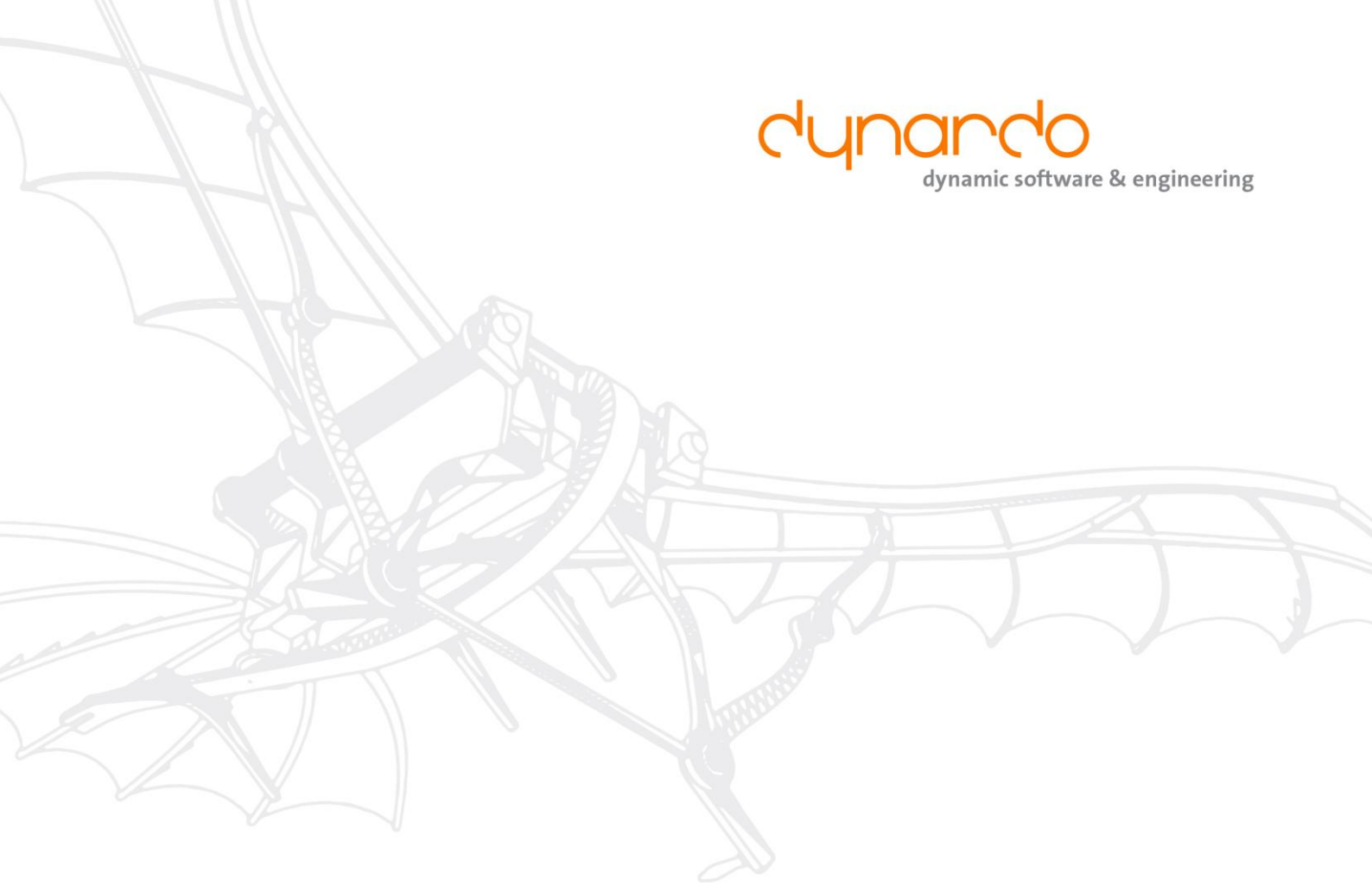

Lectures

Methods of Reliability Analysis in the context of RDO

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Methods of Reliability Analysis in the context of RDO

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Abstract

This paper shows the application of reliability methods in the content of robust design optimization. Several reliability methods are presented and discussed. The consequences of the proper treatment of stochastic uncertainties are demonstrated with the aid of a simple structural example. As a result it is seen that the suggested reliability methods help to define safety factors to compensate for unavoidable uncertainties in the design optimization process.

Keywords: RDO, structural optimization, reliability analysis, robustness analysis.

1 Introduction

Current developments in CAE very much focus on automated design improvements. This can be achieved by implementing optimization techniques to increase the performance of engineering products and systems. As it is well-known, optimization which is fine-tuned to a particular set of operating conditions may lead to designs whose performance deteriorates considerably whenever the actual operating conditions change. This is called loss of *robustness*. In order to cope with the unavoidable uncertainties in both operating conditions as well as the manufacturing process itself, it is essential to introduce appropriate robustness measures based on uncertainty analysis into the optimization process.

Due to its mature state of development it is very helpful to utilize probability theory in conjunction with statistics to describe and quantify uncertainties. Uncertainties can enter the optimization process in various forms as shown in Figs. 1.

Assume that the constraints are expressed in terms of functions g. The may be interpreted e.g. as safety margins which have to be positive with a sufficiently high probability. The uncertainties of the constraints (which are typically the relevant uncertainties in structural applications) can be done by introducing conditions on

- characteristic values, typically chosen as $\bar{g} + k\sigma_g > 0$, or
- failure probability, i.e. $\mathbf{Prob}[g \leq 0] < P_{acc}$

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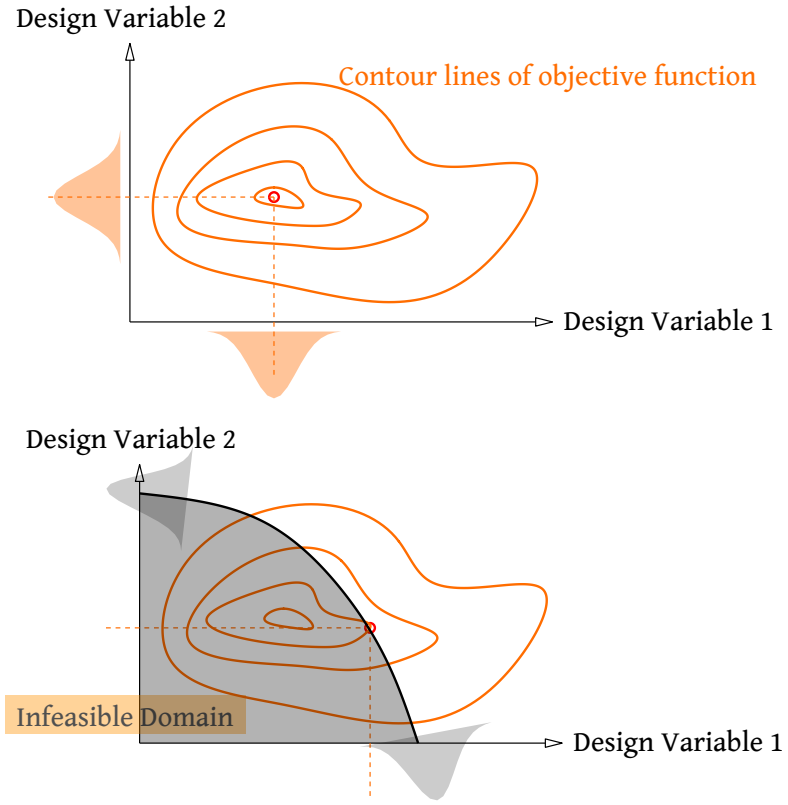

Figure 1: **Uncertainties in the optimization process, top: Manufacturing tolerances, bottom: Uncertain operating conditions affecting constraint.**

Here either the factor k or the acceptable failure probability P_{acc} must be chosen in order to satisfy the safety requirements for the structure or system under investigation.

The following sections on probabilistic modeling and reliability theory closely follow Bucher 2009a which contains a large body of additional material on the topics.

2 Probabilistic Modeling

2.1 Random vectors

In typical engineering applications the number of random variables can be fairly large. It is conceptually helpful to assemble all these random variables $X_k;\ k = 1 \ldots n$ into a *random vector* $\mathbf{X}$:

$$\mathbf{X} = [X_1, X_2, \ldots X_n]^T \tag{1}$$

For this vector, expected values can be defined in terms of expected values of all its components. In this way, we obtain the *mean value vector*

$$\bar{\mathbf{X}} = \mathbf{E}[\mathbf{X}] = [\bar{X}_1, \bar{X}_2, \ldots \bar{X}_n]^T \tag{2}$$

This definition applies the expectation operator (ensemble average) to each component of $\mathbf{X}$ individually. The *covariance matrix* is defined by

$$\mathbf{C}_{\mathbf{XX}} = \mathbf{E}[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^T] \tag{3}$$

This definition means that the expectation operator is applied to all possible mixed products of the zero mean components $(X_i - \bar{X}_i)(X_k - \bar{X}_k)$. As a consequence of this definition, the covariance matrix $\mathbf{C_{XX}}$ is obviously symmetric. In addition, it is non-negative definite (i.e. it does not have any negative eigenvalues). Therefore, it can be factored in terms of a Cholesky-decomposition

$$\mathbf{C_{XX}} = \mathbf{L}\mathbf{L}^{\mathbf{T}} \tag{4}$$

in which $\mathbf{L}$ is a non-singular lower triangular matrix. The Cholesky factor $\mathbf{L}$ can be utilized for a representation of the random variables X_i in terms of zero-mean uncorrelated random variables Y_i by applying a linear transfomation:

$$\mathbf{Y} = \mathbf{L}^{-1}(\mathbf{X} - \bar{\mathbf{X}}); \quad \mathbf{X} = \mathbf{L}\mathbf{Y} + \bar{\mathbf{X}} \tag{5}$$

We can easily prove that the mean value vector of $\mathbf{Y}$ is zero

$$\mathbf{E}[\mathbf{Y}] = \mathbf{E}[\mathbf{L}^{-1}(\mathbf{X} - \bar{\mathbf{X}})] = \mathbf{L}^{-1}\mathbf{E}[\mathbf{X} - \bar{\mathbf{X}}] = 0 \tag{6}$$

and that the components of $\mathbf{Y}$ are uncorrelated:

$$\begin{aligned} \mathbf{E}[\mathbf{Y}\mathbf{Y}^T] = \mathbf{C}_{YY} &= \mathbf{E}[\mathbf{L}^{-1}(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^T\mathbf{L}^{-1T}] \\ &= \mathbf{L}^{-1}\mathbf{L}\mathbf{L}^T\mathbf{L}^{-1T} = \mathbf{I} \\ &\rightarrow \mathbf{E}[Y_i^2] = 1 \quad \forall i; \quad \mathbf{E}[Y_i Y_k] = 0 \quad \forall i \neq k \end{aligned} \tag{7}$$

The dimensionless quantity

$$\rho_{ik} = \frac{\mathbf{E}[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]}{\sigma_{X_i}\sigma_{X_k}} \tag{8}$$

is called *coefficient of correlation*. Its value is bounded in the interval $[-1,\ 1]$. Note that the matrix of correlation coefficients must be positive definite as well. This poses certain restrictions on the numerical values of ρ_{ik} depending on the dimension of the random vector $\mathbf{X}$.

2.2 Joint probability density function models

Multi-dimensional Gaussian distribution

The pdf of jointly normally (Gaussian) distributed random variables (components of a random vector $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{C_{XX}}}} \exp\left[-\frac{1}{2}(\mathbf{x} - \bar{\mathbf{X}})^T\mathbf{C}_{\mathbf{XX}}^{-1}(\mathbf{x} - \bar{\mathbf{X}})\right];\ \mathbf{x} \in \mathbb{R}^n \tag{9}$$

Independent random variables

If all random variables X_i are mutually independent, then the joint probability density function is given by the product of the individual probability density functions.

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} f_{X_i}(x_i) \tag{10}$$

This follows from the multiplication rule for independent events.

Remarks:

1. Independent random variables are always uncorrelated. The reverse is not necessarily true.
2. If the random variables $X_i, i = 1 \ldots n$ are jointly normally distributed and they are pairwise uncorrelated, then they are pairwise independent.

Nataf-Model

The so-called *Nataf*-model (Nataf 1962; Liu and DerKiureghian 1986) describes the joint probability density function of random variables X_i based on their individual (marginal) distributions and the covariances or coefficients of correlation ρ_{ik}. The concept of this Gaussian copula (Noh, Choi, and Du 2009) is to transform the original variables X_i to Gaussian variables Y_i whose joint density is assumed to be multi-dimensional Gaussian. This model can be realized in three steps:

1. Map all random variables X_i individually to normally distributed random variables V_i with zero mean and unit standard deviation

$$\{X_i; f_{X_i}(x_i)\} \leftrightarrow \{V_i; \varphi(v_i)\} \tag{11}$$

   which is accomplished by means of

$$V_i = \Phi^{-1}[F_{X_i}(X_i)] \tag{12}$$

2. Assume a jointly normal distribution for all random variables V_i with the statistical moments

$$\mathbf{E}[V_i] = 0; \quad \mathbf{E}[V_i^2] = 1; \quad \mathbf{E}[V_i V_k] = \rho'_{ik} \tag{13}$$

   Note that at this point, the correlation coefficient ρ'_{ik} (which generally will be different from ρ_{ik}) is not yet known. The joint pdf for the components of the random vector $\mathbf{V}$ is then

$$f_{\mathbf{V}}(\mathbf{v}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{R}_{\mathbf{VV}}}} \exp\left(-\frac{1}{2}\mathbf{v}^T\mathbf{R}_{\mathbf{VV}}^{-1}\mathbf{v}\right) \tag{14}$$

   in which $\mathbf{R}_{\mathbf{VV}}$ denotes the matrix of all correlations ρ'_{ik}. From this relation, it follows that

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{V}}[\mathbf{v}(\mathbf{x})] \prod_{i=1}^{n} \left|\frac{dx_i}{dv_i}\right| = f_{\mathbf{V}}[\mathbf{v}(\mathbf{x})] \prod_{i=1}^{n} \frac{f_{X_i}(x_i)}{\varphi[v_i(x_i)]} \tag{15}$$

3. Compute the correlation coefficients ρ'_{ik} by solving

$$\sigma_{x_i}\sigma_{x_j}\rho_{ik} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i - \bar{X}_i)(x_k - \bar{X}_k) f_{X_i X_k}(x_i, x_k, \rho'_{ik})\, x_i x_k \tag{16}$$

   This is usually achieved by iteration.

A known problem of the Nataf-model is that this iteration may lead to a non-positive-definite matrix of correlation coefficients. In this case, this model is not applicable. A set of semi-empirical formulas relating ρ and ρ' based on numerical studies for various types of random variables is given by Liu and DerKiureghian 1986.

Example: two log-normally distributed random variables

Consider two log-normally distributed random variables X_1, X_2 with identical means $\bar{X}_1 = \bar{X}_2 = \bar{X} = 1$ and identical standard deviations $\sigma_1 = \sigma_2 = \sigma = 0.4$. Assume further that the two variables are correlated with a coefficient of correlation ρ. The individual variables are readily mapped to standard Gaussian variables V_1, V_2 by means of

$$V_i = \Phi^{-1}[F_{X_i}(X_i)] = \Phi^{-1}\Phi\left(\frac{\log\frac{X_i}{\mu}}{s}\right) = \frac{1}{s}\log\frac{X_i}{\mu} = \frac{1}{s}\log\left(\frac{X_i\sqrt{\bar{X}^2+\sigma^2}}{\bar{X}^2}\right) \tag{17}$$

For the numerical values as given, this reduces to

$$V_i = 2.5957 \cdot \log(1.077 \cdot X_i); \quad X_i = 0.92848 \cdot \exp(0.38525 V_i) \tag{18}$$

The joint pdf can be derived as shown, e.g. in Noh, Choi, and Du 2009:

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi s_1 s_2\sqrt{1-\rho'^2}x_1x_2} \cdot \exp\left[-\frac{v_1^2 - 2\rho' v_1 v_2 + v_2^2}{2(1-\rho'^2)}\right] \tag{19}$$

in which

$$v_i = \frac{\log x_i - \log\mu_i}{s_i}; \quad i = 1,2 \tag{20}$$

The coefficient of correlation ρ' in $\mathbf{v}$-space according to Eq. 16 is to be determined from

$$0.16 \cdot \rho = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-1)(x_2-1)f_{X_1X_2}(x_1,x_2)x_1x_2 \tag{21}$$

An explicit solution is given by Noh, Choi, and Du 2009

$$\rho' = \frac{\log(1+\rho s_1 s_2)}{\sqrt{\log(1+s_1^2)\log(1+s_2^2)}} \tag{22}$$

This relation is shown in Fig. 2 for $s_1 = s_2 = 0.5$. It can be seen that the difference between ρ and ρ' is very small for positive values. However, when ρ approaches the lower limit of -1, there is no acceptable solution for ρ' leading to a positive definite correlation matrix. This effect is further increased by larger values of s_1, s_2. Hence, the Nataf model ceases to function properly in this range. For a more specific example, assume a coefficient of correlation $\rho = 0.6$. In this case, the correlation coefficient in Gaussian space according to Eq. 22 becomes ρ' = 0.6147. The contour lines of the joint probability density function are shown in Fig. 3. Also shown in the same figure is the case of one variable chosen normal and the second variable assumed log-normal.

3 Structural Reliability Methods

3.1 Definitions

Generally, failure (i.e. an undesired or unsafe state of the structure) is defined in terms of a limit state function $g(.)$, i.e. by the set $\mathcal{F} = \{\mathbf{X} : g(\mathbf{X})0\}$. Frequently, $Z = g(\mathbf{X})$ is called *safety margin*.

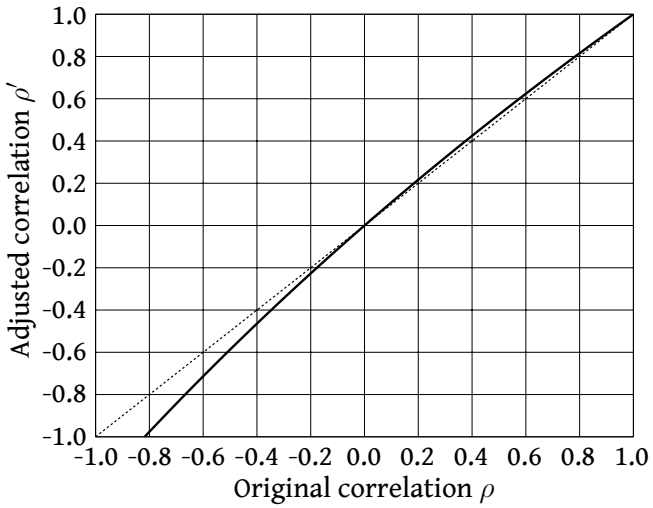

Figure 2: Relation between original and adjusted correlation for two correlated log-normal variables using the Nataf model

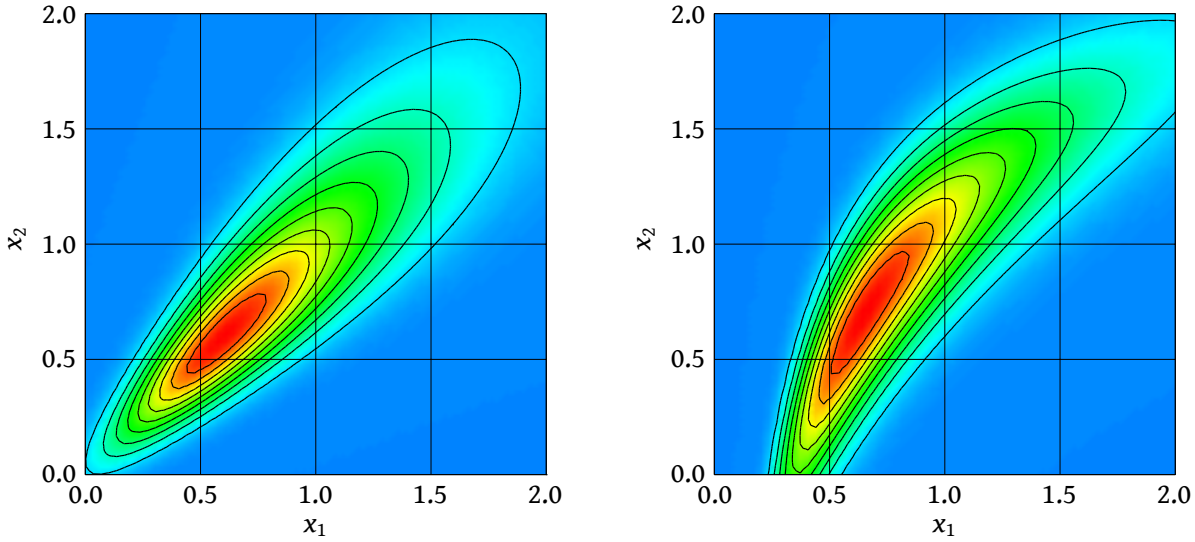

Figure 3: Contour lines of joint probability density function for correlated random variables based on the Nataf model. left: two lognormal variable; right: one normal, one lognormal variable.

For the simple problem as shown in Fig. 4, the definition of the limit state function is not unique, i.e. there are several ways of expressing the failure condition

$$\mathcal{F} = \{(F, L, M_{pl}) : FLM_{pl}\} = \{(F, L, M_{pl}) : 1 - \frac{FL}{M_{pl}} 0\} \tag{23}$$

The failure probability is defined as the probability of the occurrence of $\mathcal{F}$:

$$p_f = \mathbf{Prob}[\{\mathbf{X} : g(\mathbf{X}) 0\}] \tag{24}$$

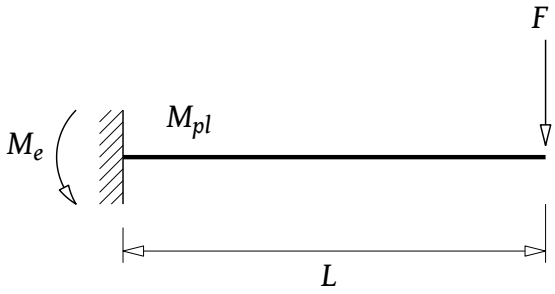

Figure 4: Simple structural system

This quantity is *unique*, i.e. not depending on the particular choice of the limit state function. The failure probability can be written in the form of a multi-dimensional integral

$$p_f = \mathbf{Prob}[g(X_1, X_2, \ldots X_n)0] = \int_{g(\mathbf{X})\leq 0} \cdots \int f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{25}$$

The computational challenge in determining the integral of Eq. 25 lies in evaluating the limit state function $g(\mathbf{x})$, which for non-linear systems usually requires an incremental/iterative numerical approach. In this context, it is essential to realize that the limit state function $g(\mathbf{x})$ serves the sole purpose of defining the bounds of integration in Eq. (25). As an example, consider a 2-dimensional problem with standard normal random variables X_1 and X_2, and a limit state function $g(x_1, x_2) = 3 - x_1 + x_2$. In Figure 5 the integrand of Eq. 25 in the failure domain is displayed. It is clearly visible that only a very narrow region around the so-called design point $\mathbf{x}^*$ really contributes to the value of the integral, i.e., the probability of failure $P(\mathcal{F})$. This makes is difficult to locate integration points for numerical integration procedures appropriately.

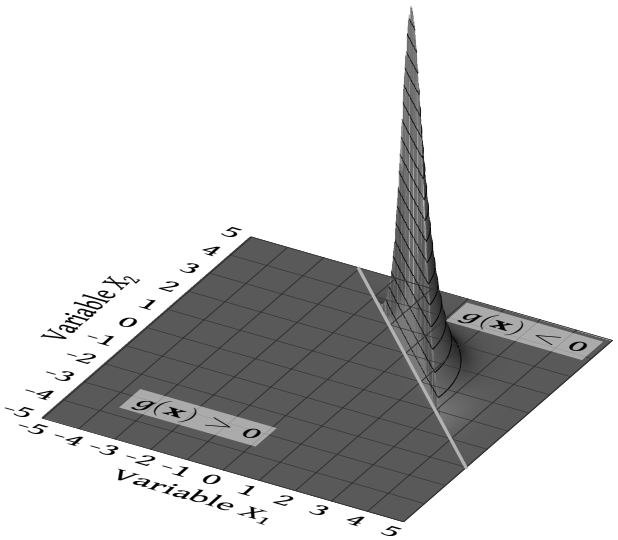

Figure 5: Integrand for calculating the probability of failure for $g(x_1, x_2) = 3 - x_1 - x_2$

3.2 First Order - Second Moment Concept

The first-order second moment method aims at a representation of the limit state function $g(.)$ by a Taylor series and subsequent calculation of the statistical moments of the safety margin Z.

$$\begin{aligned} g(\mathbf{x}) = g(\mathbf{x}_0) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_0} (x_i - x_{i0}) + \\ + \frac{1}{2} \sum_{i=1}^{n}\sum_{k=1}^{n} \left.\frac{\partial^2 g}{\partial x_i \partial x_k}\right|_{\mathbf{x}=\mathbf{x}_0} (x_i - x_{i0})(x_k - x_{k0}) + \dots \end{aligned} \tag{26}$$

Terminating the series after the *quadratic* terms yields

$$\mathbf{E}[Z] = \mathbf{E}[g(\mathbf{X})] = g(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\mathbf{E}[x_i - x_{i0}] + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_k}\mathbf{E}[(x_i - x_{i0})(x_k - x_{k0})] \tag{27}$$

Terminating the Taylor series after the *linear* terms yields

$$\mathbf{E}[Z] = \mathbf{E}[g(\mathbf{X})] = g(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\mathbf{E}[x_i - x_{i0}] \tag{28}$$

If the mean value vector $\bar{\mathbf{X}}$ is chosen as expansion point $\mathbf{x}_0$ for the Taylor series, then $E[Z] = g(\mathbf{x}_0)$ and the variance becomes

$$\begin{aligned} \sigma_Z^2 = \mathbf{E}[(Z - \bar{Z})^2] &= \mathbf{E}\left[\left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X_i - \bar{X}_i)\right)^2\right] \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_k}\mathbf{E}[(X_i - \bar{X}_i)(X_k - \bar{X}_k)] \end{aligned} \tag{29}$$

Finally the distribution function $F_Z(z)$ is approximated by a normal distribution

$$F_Z(z) = \Phi\left(\frac{z - \bar{Z}}{\sigma_Z}\right) \tag{30}$$

Then we obtain the approximate result

$$p_f = F_Z(0) = \Phi\left(-\frac{\bar{Z}}{\sigma_Z}\right) \tag{31}$$

Note that this result does not take into account the types of distributions of the basic variables. It also depends significantly on the choice of the expansion point for the Taylor-series.

3.3 FORM - First Order Reliability Method

The FORM-Concept (Hasofer and Lind 1974) is based on a description of the reliability problem in standard Gaussian space. Hence transformations from correlated non-Gaussian variables $\mathbf{X}$ to uncorrelated Gaussian variables $\mathbf{U}$ with zero mean and unit variance are required.

This step is called Rosenblatt-transformation. Then a linearization in performed in $\mathbf{u}$-space. The expansion point $\mathbf{u}^*$ is chosen such as to maximize the pdf within the failure domain. Geometrically, this coincides with the point in the failure domain, having the minimum distance β from the origin. From a safety engineering point of view, the point $\mathbf{x}^*$ corresponding to $\mathbf{u}^*$ is called *design point*.

This concept is especially useful in conjunction with the Nataf-model for the joint pdf of $\mathbf{X}$. In this case the Rosenblatt-transformation consists of the following steps:

1. Transform from correlated non-Gaussian variables X_i to correlated Gaussian variables Y_i

$$Y_i = \Phi^{-1}[F_{X_i}(X_i)]; \quad i = 1n \tag{32}$$

   These transformations can be carried out independently. The covariance matrix $\mathbf{C_{YY}}$ is calculated from $\mathbf{C_{XX}}$ according to the rules of the Nataf-model (cf. section 2.2).

2. Transform from correlated Gaussian space to standard Gaussian space by means of

$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{Y} \tag{33}$$

   in which $\mathbf{L}$ is calculated from the Cholesky-decomposition of $\mathbf{C_{YY}}$

$$\mathbf{C_{YY}} = \mathbf{L}\mathbf{L}^T \tag{34}$$

In total, this leads to a representation of the limit state function $g(.)$ in terms of the standardized Gaussian variables U_i

$$g(\mathbf{X}) = g(X_1, X_2, X_n) = g[X_1(U_1, U_n)X_n(U_1, U_n)] \tag{35}$$

with

$$X_i = F_{X_i}^{-1}\left[\Phi\left(\sum_{k=1}^{n} L_{ik}U_k\right)\right] \tag{36}$$

From the geometrical interpretation of the expansion point $\mathbf{u}^*$ in standard Gaussian space it becomes quite clear that the calculation of the design point can be reduced to an optimization problem

$$\mathbf{u}^* = \text{argmin}\left(\frac{1}{2}\mathbf{u}^T\mathbf{u}\right); \quad \text{subject to: } g[\mathbf{x}(\mathbf{u})] = 0 \tag{37}$$

This leads to the Lagrange-function

$$L = \frac{1}{2}\mathbf{u}^T\mathbf{u} + \lambda g(\mathbf{u}) \rightarrow \text{Min.} \tag{38}$$

Standard optimization procedures can be utilized to solve for the location of $\mathbf{u}^*$. One of the earliest methods is the so-called Rackwith-Fiessler algorithm (Rackwitz and Fiessler 1978). This algorithm is a simple version of the SQP optimization procedure. In this procedure, the objective function is replaced by a quadratic approximation and the constrain conditions are linearized. In view of Lagrangian as given by Eq. 38 this means that the objective function is unchanged whereas the constraint is replaced by the linearized version using $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$

$$\hat{g}(\mathbf{u}) = g(\mathbf{u}_0) + \nabla g(\mathbf{u}_0)^T(\mathbf{u} - \mathbf{u}_0) = g(\mathbf{u}_0) + \nabla g(\mathbf{u}_0)^T\mathbf{v} \tag{39}$$

In this equation, $\mathbf{u}_0$ is an expansion point, usually chosen to be the current iterate. The approximate Lagrangian

$$\begin{aligned}\hat{L} &= \frac{1}{2}\mathbf{u}^T\mathbf{u} + \lambda\hat{g}(\mathbf{u}) \\ &= \frac{1}{2}\mathbf{v}^T\mathbf{v} + \mathbf{u}_0^T\mathbf{v} + \frac{1}{2}\mathbf{u}_0^T\mathbf{u}_0 + \lambda\left[g(\mathbf{u}_0) + \nabla g(\mathbf{u}_0)^T\mathbf{v}\right]\end{aligned} \tag{40}$$

is associated with the Kuhn-Tucker conditions

$$\begin{aligned}\mathbf{v} + \mathbf{u}_0 + \lambda\nabla g(\mathbf{u}_0) &= 0 \\ g(\mathbf{u}_0) + \nabla g(\mathbf{u}_0)^T\mathbf{v} &= 0\end{aligned} \tag{41}$$

This system of equations is solved by

$$\lambda = \frac{g(\mathbf{u}_0) - \nabla g(\mathbf{u}_0)^T\mathbf{u}_0}{\nabla g(\mathbf{u}_0)^T\nabla g(\mathbf{u}_0)} \tag{42}$$

and

$$\mathbf{u}_0 + \mathbf{v} = -\lambda\nabla g(\mathbf{u}_0) \tag{43}$$

Then $\mathbf{u}_0$ is replaced by $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$ and the iteration proceeds from Eq. 39 until convergence of $\mathbf{u}_0$ to $\mathbf{u}^*$. It is known that this simple version of the algorithm does not always converge, hence more sophisticated optimization methods may be appropriate (e.g. NLPQL, Schittkowski 1986).

Once the point $\mathbf{u}^*$ is located, the exact limit state function $g(\mathbf{u})$ is replaced by a linear approximation $\hat{g}(\mathbf{u})$ as shown in Fig. 6. Geometrically, it can easily be seen that $\hat{g}(\mathbf{u})$ is

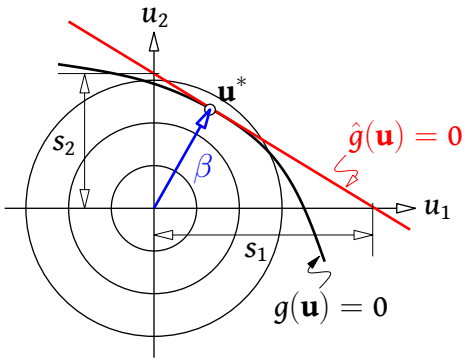

Figure 6: Linearization required for first order reliability method

determined from

$$\hat{g} : -\sum_{i=1}^{n}\frac{u_i}{s_i} + 1 = 0; \quad \sum_{i=1}^{n}\frac{1}{s_i^2} = \frac{1}{\beta^2} \tag{44}$$

The safety margin $Z = -\sum_{i=1}^{n}\frac{U_i}{s_i} + 1$ is normally distributed with the following statistical moments

$$E[Z] = 1; \quad \sigma_Z^2 = \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{E[U_iU_k]}{s_is_k} = \sum_{i=1}^{n}\frac{E[U_i^2]}{s_i^2} = \sum_{i=1}^{n}\frac{1}{s_i^2} = \frac{1}{\beta^2} \tag{45}$$

$$\rightarrow \sigma_Z = \frac{1}{\beta} \tag{46}$$

From this, the probability of failure is easily determined to be

$$p_f = \Phi\left(-\frac{1}{\frac{1}{\beta}}\right) = \Phi(-\beta) \tag{47}$$

This result is exact, if $g(\mathbf{u})$ is actually linear.

4 Monte Carlo Simulation

4.1 Definitions and Basics

The definition of the failure probability as given in Eq. 25 can be written as an expected value

$$p_f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} I_g(x_1 \dots x_n) f_{X_1 \dots X_n}(x_1 \dots x_n) dx_1 \dots dx_n \tag{48}$$

in which $I_g(x_1 \dots x_n) = 1$ if $g(x_1 \dots x_n) \leq 0$ and $I_g(.) = 0$ else.

In order to determine p_f in principle all available statistical methods for estimation of expected values are applicable. If m independent samples $\mathbf{x}^{(k)}$ of the random vector $\mathbf{X}$ are available then the estimator

$$\bar{p}_f = \frac{1}{m}\sum_{k=1}^{m} I_g(\mathbf{x}^{(k)}) \tag{49}$$

yields a consistent and unbiased estimate for p_f.

The problem associated with this approach is this: For small values of p_f and small values of m the confidence of the estimate is very low. The variance $\sigma^2_{\bar{p}_f}$ of the estimate $\bar{p}_f$ can be determined from

$$\sigma^2_{\bar{p}_f} = \frac{p_f}{m} - \frac{p_f^2}{m} \approx \frac{p_f}{m} \rightarrow \sigma_{\bar{p}_f} = \sqrt{\frac{p_f}{m}} \tag{50}$$

It is to be noted that the required number m of simulations is independent of the dimension n of the problem!

4.2 Importance Sampling (Weighted Simulation)

4.2.1 General Concept

In order to reduce the standard deviation $\sigma_{\bar{p}_f}$ of the estimator to the order of magnitude of the probability of failure itself m must be in the range of $m = \frac{1}{p_f}$. For values of p_f in the range of 10^{-6} this cannot be achieved if each evaluation of the limit state function requires a complex structural analysis. Alternatively, strategies are employed which increase the "hit-rate" by artificially producing more samples in the failure domain than should occur according to the distribution functions. One way to approach this solution is the introduction of a positive weighting function $h_{\mathbf{Y}}(\mathbf{x})$ which can be interpreted as density function of a random vector $\mathbf{Y}$. Samples are taken according to $h_{\mathbf{Y}}(\mathbf{x})$.

The probability of failure is then estimated from

$$\bar{p}_f = \frac{1}{m}\sum_{k=1}^{m}\frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})}I_g(\mathbf{x}) = \mathbf{E}\left[\frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})}I_g(\mathbf{x})\right] \tag{51}$$

From the estimation procedure as outlined it can be seen that the variance of the estimator $\bar{p}_f$ becomes

$$\sigma^2_{\bar{p}_f} = \frac{1}{m}\mathbf{E}\left[\frac{f_{\mathbf{X}}(\mathbf{x})^2}{h_{\mathbf{Y}}(\mathbf{x})^2}I_g(\mathbf{x})\right] \tag{52}$$

A useful choice of $h_{\mathbf{Y}}(\mathbf{x})$ should be based on minimizing $\sigma^2_{\bar{p}_f}$. Ideally, the weighting function should reduce the sampling error to zero. However, this cannot be achieved in reality since such a function must have the property

$$h_{\mathbf{Y}}(\mathbf{x}) = \begin{cases} \frac{1}{p_f}f_{\mathbf{X}}(\mathbf{x}) & g(\mathbf{x})0 \\ 0 & g(\mathbf{x}) > 0 \end{cases} \tag{53}$$

This property requires the knowledge of p_f which, of course, is unknown. Special updating procedures such as adaptive sampling (Bucher 1988) can help to alleviate this problem.

4.2.2 Importance Sampling at the Design Point

Based on the previous FORM analysis it may be attempted to obtain a general importance sampling concept. This can be accomplished in two steps:

1. Determine the design point $\mathbf{x}^*$ as shown in the context of the FORM-procedure.
2. Choose a weighting function (sampling density) $h_{\mathbf{Y}}(\mathbf{x})$ with the statistical moments $E[\mathbf{Y}] = \mathbf{x}^*$ and $\mathbf{C}_{\mathbf{YY}} = \mathbf{C}_{\mathbf{XX}}$ in the following form (multi-dimensional Gaussian distribution, cf. Fig. 7)

$$h_{\mathbf{Y}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{C}_{\mathbf{XX}}}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{x}^*)^T\mathbf{C}^{-1}_{\mathbf{XX}}(\mathbf{x}-\mathbf{x}^*)\right] \tag{54}$$

3. Perform random sampling and statistical estimation according to Eq. 51.

The efficiency of this concept depends on the geometrical shape of the limit state function. In particular, limit state functions with high curvatures or almost circular shapes cannot be covered very well.

It is also interesting to note that the concept of importance sampling can very well be extended for application in the context of dynamic problems (first passage failure, Macke and Bucher 2003).

4.2.3 Adaptive Sampling

As mentioned earlier, the "optimal" sampling density should satisfy the requirement

$$h_{\mathbf{Y}}(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x}|\mathbf{x} \in D_f) \tag{55}$$

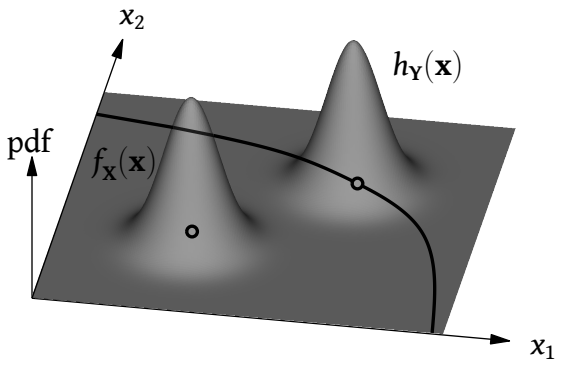

Figure 7: Original and importance sampling probability density functions

Here the failure domain D_f is the set in which the limit state function is negative

$$D_f = \{\mathbf{x}|g(\mathbf{x})0\} \tag{56}$$

This ideal condition cannot be met strictly. Yet it is possible to meet it in a second moment sense, i.e. $h_{\mathbf{Y}}(\mathbf{x})$ can be chosen such that (Bucher 1988)

$$\mathbf{E}[\mathbf{Y}] = \mathbf{E}[\mathbf{X}|\mathbf{X} \in D_f] \tag{57}$$

$$\mathbf{E}[\mathbf{Y}\mathbf{Y}^T] = \mathbf{E}[\mathbf{X}\mathbf{X}^T|\mathbf{X} \in D_f] \tag{58}$$

In terms of these statistical moments, a multi-dimensional Gaussian distribution is uniquely determined.

While, of course, this approach cannot be applied from the beginning without prior knowledge, it is fairly easy to estimate $\mathbf{E}[\mathbf{X}|\mathbf{X} \in D_f]$ and $\mathbf{E}[\mathbf{X}\mathbf{X}^T|\mathbf{X} \in D_f]$ from a pilot simulation (based on e.g. the knowledge of the design point or by using an increased sampling standard deviation to increase the number of samples in the failure domain), and then to adapt the sampling density according to these results.

For a one-dimensional reliability problem with a limit state function $g(x) = \beta - x$, the optimal sampling density is compared to a Gaussian sampling density with the same first and second moments as the optimal sampling density. Fig, 8 shows the results for β = 2 and β = 4. It can clearly be seen that the samples will be concentrated in the immediate vicinity of the design point located at $x = \beta$.

4.3 Directional Sampling

The basic idea is to simulate directions instead of points, and to solve analytically for the probability of failure conditional on a certain direction. The formulation is based on a representation of the limit state function in standard normal space (denoted by the random vector **U**). Each point **u** in this space is written in the form of

$$\mathbf{u} = r\mathbf{a} \tag{59}$$

in which r is the distance from the origin and **a** is a unit vector indicating the direction. This implies transformation to n-dimensional spherical coordinates.

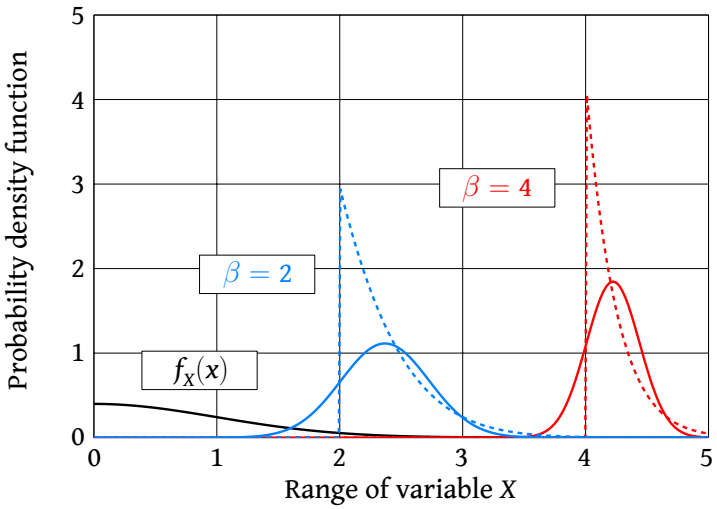

Figure 8: Optimal and best possible Gaussian sampling density functions

Density Function in Spherical Coordinates

From $f_{\mathbf{U}}(\mathbf{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{u}\right)$ we want to find the joint density $f_{R,\mathbf{A}}(r, \mathbf{a})$ of distance R and direction vector $\mathbf{A}$.

1) $f_{\mathbf{U}}(\mathbf{u})$ is rotationally symmetric, i.e. independent of $\mathbf{a}$. This follows from

$$\mathbf{u}^T\mathbf{u} = (r\mathbf{a})^T(r\mathbf{a}) = r^2\mathbf{a}^T\mathbf{a} = r^2 \tag{60}$$

and implies

$$f_{R|\mathbf{A}}(r|\mathbf{a}) = f_R(r) \tag{61}$$

This in turn yields independence of R and $\mathbf{A}$

$$f_{R,\mathbf{A}}(r, \mathbf{a}) = f_{R|\mathbf{A}}(r|\mathbf{a}) f_{\mathbf{A}}(\mathbf{a}) = f_R(r) f_{\mathbf{A}}(\mathbf{a}) \tag{62}$$

2) Due to rotational symmetry, $f_{\mathbf{A}}(\mathbf{a})$ must have identical values for any $\mathbf{a}$. Hence this density is constant. Its value is the inverse of the surface area of the n-dimensional unit sphere

$$f_{\mathbf{A}}(\mathbf{a}) = \frac{1}{S_n} = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \tag{63}$$

For $n = 2$ we have $f_A(a) = \frac{\Gamma(1)}{2\pi} = \frac{1}{2\pi}$ and for $n = 3$ we get $f_A(a) = \frac{\Gamma(1.5)}{2\pi^{\frac{3}{2}}} = \frac{1}{4\pi}$

3) The density of $r = \sqrt{\mathbf{u}^T\mathbf{u}}$ is determined from integrating the joint density of the components of $\mathbf{u}$ over a sphere with radius r leading to

$$f_R(r) = S_n r^{n-1} \frac{1}{\pi^{\frac{n}{2}}} \exp\left(-\frac{r^2}{2}\right) \tag{64}$$

For the case $n = 2$ we obtain the density function

$$f_R(r) = r \exp\left(-\frac{r^2}{2}\right) \tag{65}$$

which describes a Rayleigh distribution.

Probability of Failure

The failure probability $P(\mathcal{F}|\mathbf{a})$ conditional on a realization of the direction $\mathbf{a}$ can be determined analytically

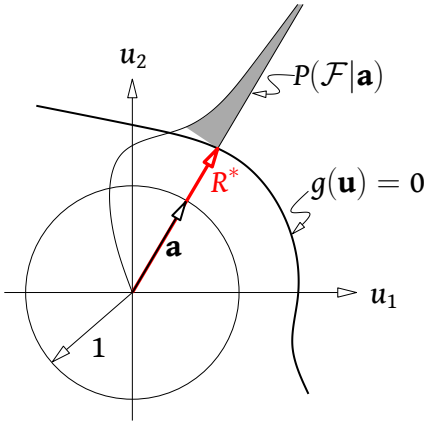

Figure 9: Directional Sampling

$$\begin{aligned} P(\mathcal{F}|\mathbf{a}) &= \int_{R^*(\mathbf{a})}^{\infty} f_{R|\mathbf{A}}(r|\mathbf{a})dr = \\ &= S_n r^{n-1} \frac{1}{\pi^{\frac{n}{2}}} \exp\left(-\frac{r^2}{2}\right) dr = 1 - \chi_n^2[R^*(\mathbf{a})^2] \end{aligned} \tag{66}$$

This is the cumulative Chi-Square-Distribution with n degrees of freedom.

Simulation Procedure

The directional sampling procedure can be summarized as follows:

1. Generate a sample $\mathbf{u}_k$ according to an n-dimensional standard normal distribution.
2. Calculate the direction vector $\mathbf{a}_k = \frac{\mathbf{u}_k}{||\mathbf{u}_k||}$.
3. Calculate critical distance $R^*(\mathbf{a}_k)$ by solving $g[R^*(\mathbf{a}_k)\mathbf{a}] = 0$. This step may involve substantial computational effort since here the limit state function must be evaluated several time, e.g. within a bisection procedure.
4. Determine conditional failure probability $P(\mathcal{F}|\mathbf{a}) = 1 - \chi_n^2[R^*(\mathbf{a}_k)^2]$.
5. Repeat above steps with $k \to k + 1$.

The method works optimally if the limit state function is circular. In this special case, one single sample yields the exact result.

4.4 Asymptotic Sampling

This relatively new approach has been presented in (Bucher 2009b; Bucher 2009a). It relies on the asymptotic behavior of the failure probability in n-dimensional i.i.d Gaussian space as the standard deviation σ of the variables and hence the failure probability P_F approaches zero (see Breitung 1984). Consider a (possibly highly nonlinear) limit state function $g(\mathbf{X})$ in which $g < 0$ denotes failure. Let σ be the standard deviation of the i.i.d. Gaussian variables $X_k, k = 1 \dots n$. We are going to determine the functional dependence of the generalized safety index β on the standard deviation σ by using an appropriate sampling technique. This is aided by some analytical considerations involving limit cases.

First, we study the case of a linear limit state function. This problem can always be reduced to a single variable by an appropriate coordinate transformation. Hence the safety index $\beta(\sigma)$ is simply given by

$$\beta(\sigma) = \frac{\beta(1)}{\sigma} \tag{67}$$

in which $\beta(1)$ is the safety index evaluated for $\sigma = 1$. Introducing the scale variable $f = \frac{1}{\sigma}$ we obtain the linear relation

$$\beta(f) = f \cdot \beta(1) \tag{68}$$

This means that in order to obtain a good estimate for $\beta(1)$, we can compute the safety index for a larger value of σ (corresponding to a smaller value of the scale f) using Monte Carlo simulation and then simply extrapolate by multiplying the obtained result with f (i.e. divide by σ).

The concept of asymptotic sampling when applied to general cases utilizes the asymptotic behavior of the safety index β by applying an extrapolation technique. Here the (assumed) functional dependence for β is chosen as

$$\beta = A \cdot f + \frac{B}{f} \tag{69}$$

This choice is motivated in order to ensure asymptotically linear behavior as $f \to \infty$ (which is equivalent to the liming case $\sigma \to 0$). The coefficients A and B are conveniently determined from a least-squares fit using Monte Carlo estimates of β for different values of f (typically for values of $f < 1$) as support points. For this fitting process, Eq. 69 is rewritten in terms of a scaled safety index as

$$\frac{\beta}{f} = A + \frac{B}{f^2} \tag{70}$$

This is illustrated qualitatively in Fig. 10.

One major advantage of this approach is its independence of the dimensionality. The accuracy is governed only by the relation between the number of samples and the probability of failure as well as the particular geometry of the limit state surface $g(\mathbf{u}) = 0$.

In this context it is essential to use a sampling method which provides very stable results. One obvious choice is Latin Hypercube Sampling (LHS) (Imam and Conover 1982; Florian 1992). Alternatively, pseudo-random sequences with low discrepancy (Niederreiter 1992; Sobol and Asotsky 2003) can be utilized. Algorithms for randomized Sobol sequences are discussed e.g. in Bratley and Fox 1988 and Hong and Hickernell 2003.

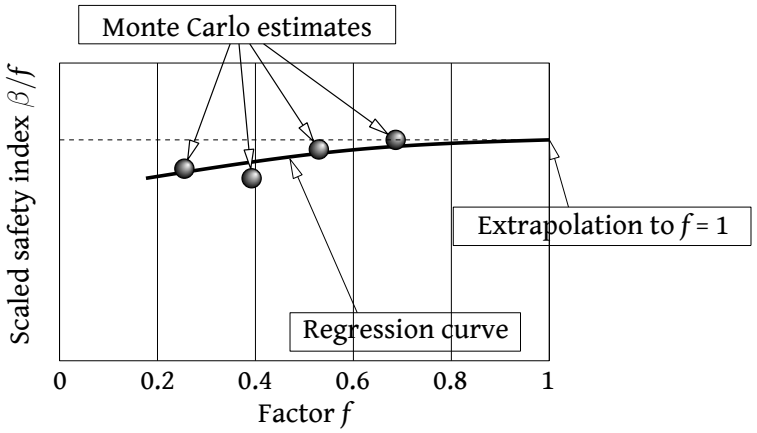

Figure 10: Basic concept of Asymptotic Sampling

High-dimensional Linear Problem

This example serves as a test case to demonstrate the independence of the dimensionality. The limit state function is

$$g(\mathbf{X}) = 5\sqrt{n} - \sum_{k=1}^{n} X_k \tag{71}$$

in which n is the number of random variables. All random variables are i.i.d. standard Gaussian. The problem has a safety index of $\beta = 5$ or $P_F = 3 \cdot 10^{-7}$, independent of n. Table 1 shows the mean values and standard deviations of the safety index as computed from asymptotic sampling (20 repetitions with 1000 Monte Carlo samples each) for different dimension n. It

Table 1: Statistics of estimated safety index for high-dimensional linear problem

n	$\hat{\beta}$	σ_β
10	4.95	0.26
100	4.94	0.22
1000	4.95	0.24
10000	4.94	0.22
100000	5.00	0.23

can be seen that the statistical uncertainty is completely independent of the dimension n, even for very large numbers of random variables.

5 RDO Example

As an example, consider a simple Euler-Bernoulli beam with rectangular cross section ($b \times h$) and length L as shown in Fig. 11. The beam is simply supported at both ends and has an additional horizontal support in mid-span. It is assumed to carry an ideally centered axial load F. There are two buckling modes, i.e. in the vertical plane and perpendicular to it. The

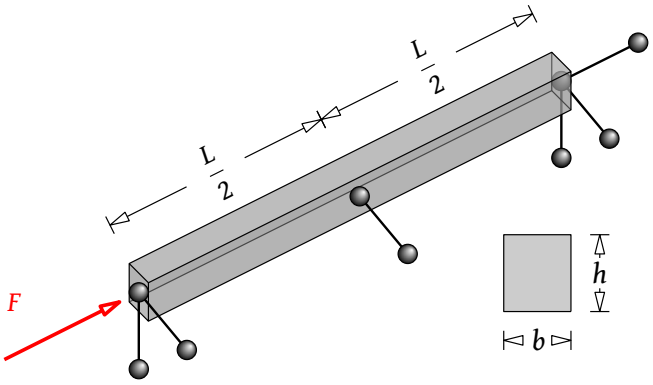

Figure 11: **Euler-Bernoulli beam under axial load**

in-plane buckling is determined by the critical load

$$F_{in} = \frac{\pi^2 b h^3}{12 L^2} \tag{72}$$

and the out-of-plane buckling is determined by the critical load

$$F_{out} = \frac{\pi^2 b^3 h}{3 L^2} \tag{73}$$

Note that out of plane the buckling length is equal to half the beam length. Assume that the total mass of the beam should be minimized under the constraint that the critical load in either mode does not become smaller than the applied load F. This constrained optimization problem is solved for the numerical values of E = 210 GPa and L = 4 m as well as F = 10 MN. The objective function together with the constraints is shown in Fig. 12. The deterministic optimum is located at the point where both critical loads are just reached, i.e. at b^* = 0.1037 m and h^* = 0.2075 m.

In the next step we assume random properties for the width b and height h of the cross section as well as the applied load F (see Table 2). The optimization now uses the mean values $\bar{b}$ and $\bar{h}$ as design variables. Depending on the probabilistic assessment method, different

Table 2: Statistical properties of random variables

Variable	Mean value	C.o.V. [%]	Distribution
b	$\bar{b}$	5	Normal
h	$\bar{h}$	5	Normal
F	10 MN	0.3	Log-normal

optimal designs $\bar{b}^*$ and $\bar{h}^*$ are obtained. The methods to describe the uncertainty of the safety margins as investigated are:

a) FOSM approach to approximate the mean value and standard deviation of the safety margins g, constraint condition $\bar{g} + 3\sigma_g > 0$.

b) Monte Carlo simulation to estimate mean value and standard deviation of the safety margins g, constraint condition $\bar{g} + 3\sigma_g > 0$.

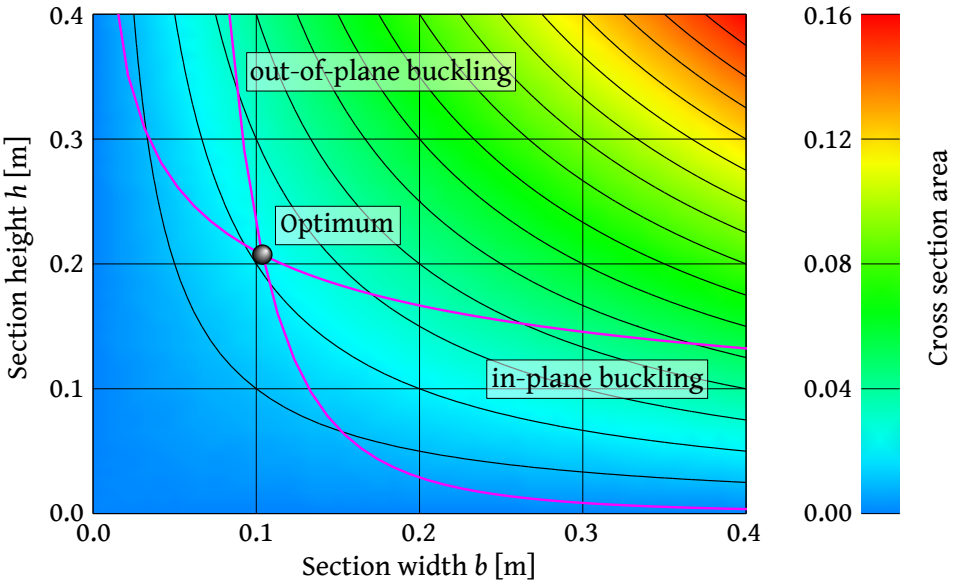

Figure 12: **Deterministic optimization problem**

c) FORM reliability analysis, constraint condition $\mathbf{Prob}[g \leq 0] < 10^{-5}$.

d) FORM reliability analysis, constraint condition $\mathbf{Prob}[g \leq 0] < 10^{-6}$.

The results from these 4 different approaches are shown in Fig. 13 and compared to the deterministic optimization result. Table 3 shows the numerical values of the optimal designs

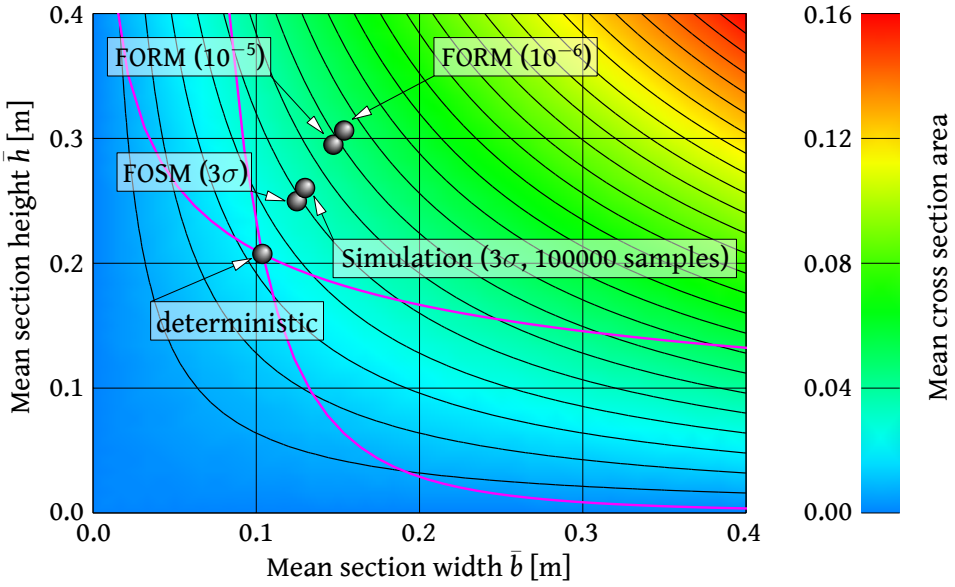

Figure 13: **Robust design optimization problem**

for these cases. It can clearly be seen that the method chosen and the level of safety applied can affect the design substantially. The increase in total weight from a deterministic deign

without any safety factors to a probabilistic design with a safety level of 0.999999 amounts to almost 220% (i.e. more than doubles the weight).

Table 3: Robust optimization results

Case	b^* [m]	h^* [m]	$\bar{A}$ [m^2]	Increase [%]
Deterministic	0.104	0.208	0.02152	100
a	0.125	0.250	0.03118	145
b	0.130	0.260	0.03381	157
c	0.147	0.295	0.04343	202
d	0.154	0.306	0.04712	219

6 Concluding Remarks

The question of uncertainties in the design optimization process naturally leads to the question of design robustness. In order to incorporate robustness analysis into the automated design process it is essential to utilize appropriate stochastic analysis. For problems involving safety constraints, usually reliability analysis is the suitable choice. Unfortunately, reliability or failure analysis is computationally very expensive. Therefore, the choice of appropriate methods is essential for the success of the RDO approach. This may include the utilization of approximations to the system response e.g. as available in the Response Surface Method.

The major benefit of using a full probabilistic RDO approach lies in the rational assessment and quantification of the ”safety factors” required to achieve a target safety level. Obviously, the best procedure is to incorporate the reliability analysis fully into the design optimization loop. In this way, robustness of the final design can be automatically guaranteed.

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