[WOST 2012, November 30, 2012]

Bayesian analysis applied to stochastic mechanics and reliability: Making the most of your data and observations

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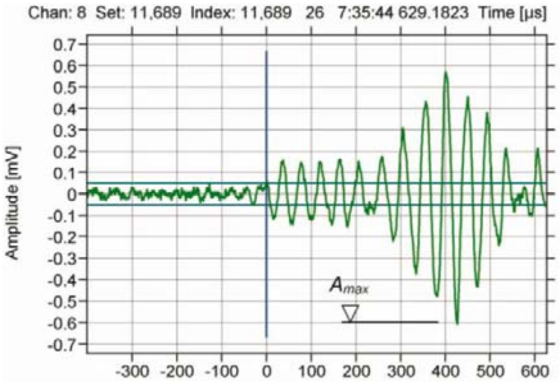
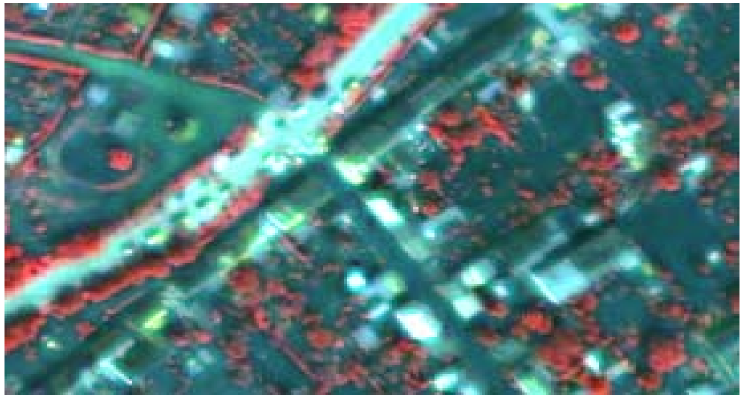
Ever increasing amounts of information are available

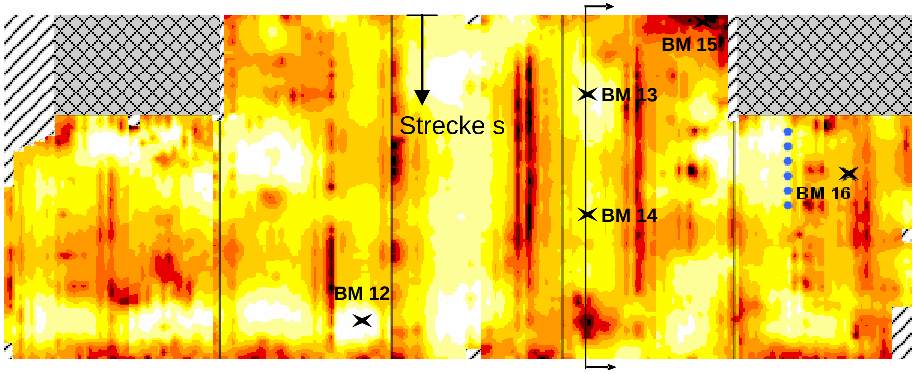
Sensor data



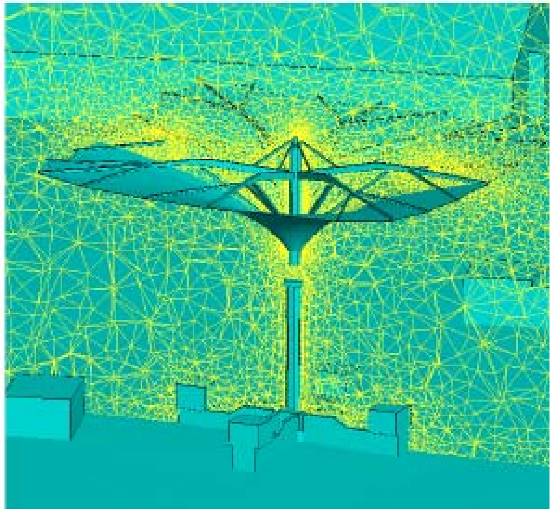
Satelite data



Spatial measurements on structures



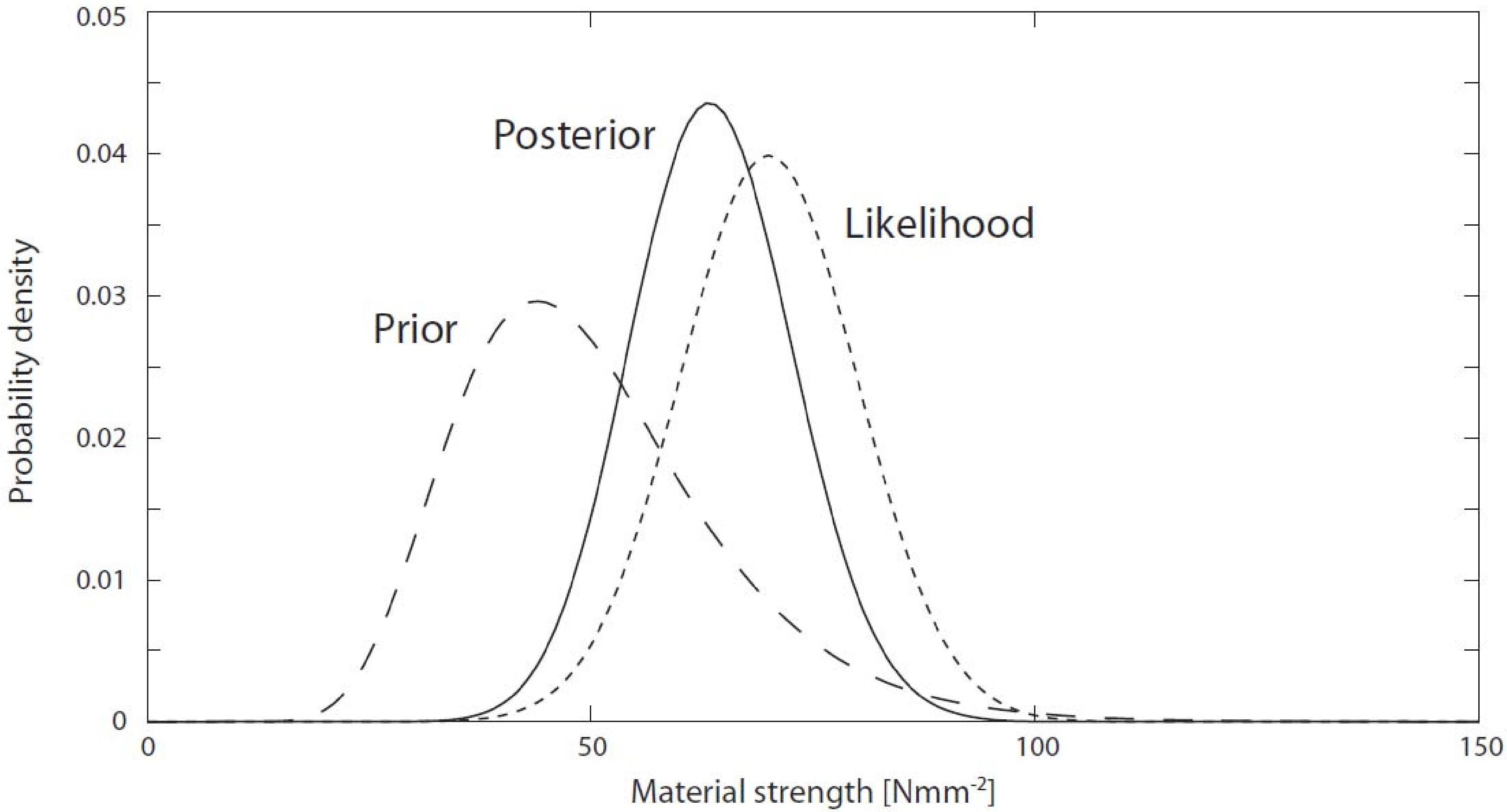
Advanced simulation



Sources: Frey et al. (in print); Gehlen et al. (2010); Michalski et al (2011); Schuhmacher et al. (2011)

Updating models and reliability computations with (indirect) information

- Bayes' rule: $Posterior \propto Prior \times Likelihood$



How to compute the reliability of a geotechnical site conditional on deformation monitoring outcomes?

-> Integrate
Bayesian updating in structural reliability methods

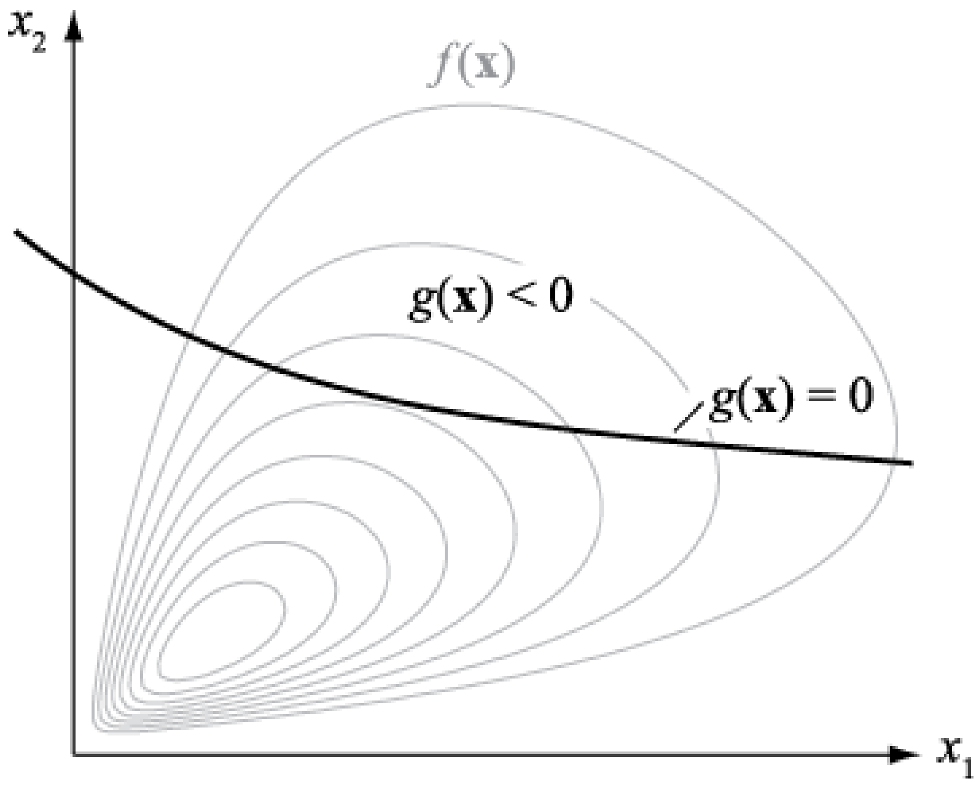
Prior model in structural reliability

- Failure domain:

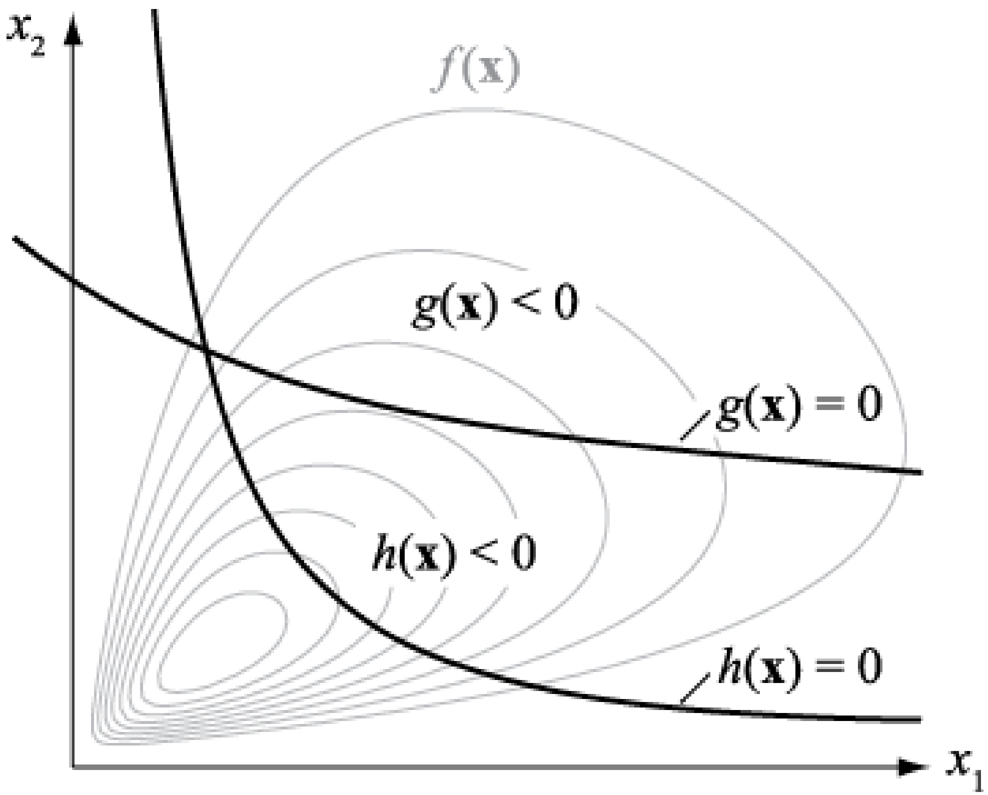
$$\Omega_F = \{g(\mathbf{x}) \leq 0\}$$

- Probability of failure:

$$\Pr(F) = \int_{\mathbf{x}\in\Omega_F} f(\mathbf{x})\mathrm{d}\mathbf{x}$$



Information in structural reliability



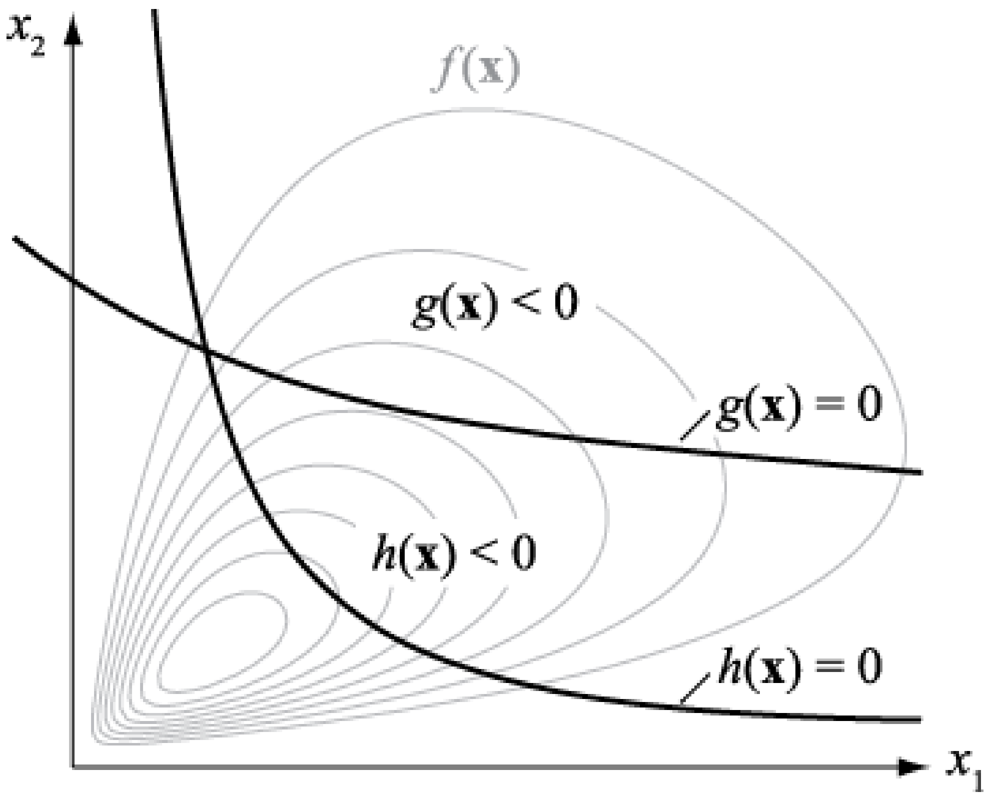
- Inequality information:

  $\Omega_Z = \{h\ (\mathbf{x}) \leq 0\}$

- Conditional probability of failure:

$$\Pr(F|Z) = \frac{\Pr(F \cap Z)}{\Pr(Z)} = \frac{\int_{\mathbf{x} \in \{\Omega_F \cap \Omega_Z\}} f(\mathbf{x}) \mathrm{d}\mathbf{x}}{\int_{\mathbf{x} \in \{\Omega_Z\}} f(\mathbf{x}) \mathrm{d}\mathbf{x}}$$

Information in structural reliability



- Equality information:

  $\Omega_Z = \{h\ (\mathbf{x}) = 0\}$

- Conditional probability of failure:

$$\Pr(F|Z) = \frac{\Pr(F \cap Z)}{\Pr(Z)} = \frac{0}{0} = ?$$

In statistics, information is expressed as likelihood function

- Likelihood function for information event *Z*: $$L(\mathbf{x}) \propto \Pr(Z|\mathbf{X} = \mathbf{x})$$

- Example:
  - Measurement of system characteristic $s(\mathbf{X})$
  - Additive measurement error ε

- Equality information: $$h(\mathbf{X}, \epsilon) = s(\mathbf{X}) - s_m + \epsilon$$

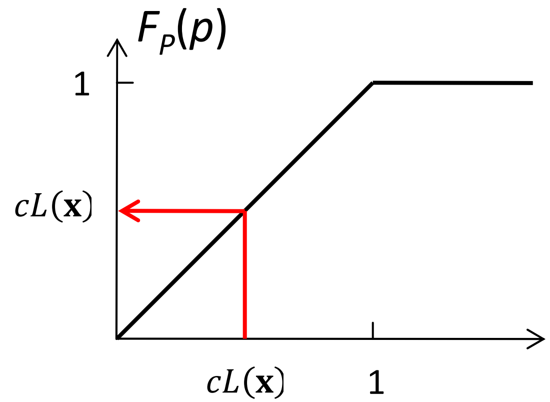
- Likelihood function: $$L(\mathbf{x}) = f_\epsilon(s_m - s(\mathbf{x})),$$

By expressing equality information as a likelihood function, it can be represented by an inequality domain

- Let
  - P be a standard uniform random variable
  - c be a constant, such that $0 \leq cL(\mathbf{x}) \leq 1$ for any $\mathbf{x}$
- then

$$cL(\mathbf{x}) = F_P[cL(\mathbf{x})]$$

$$L(\mathbf{x}) = \frac{\Pr[P \leq cL(\mathbf{x})]}{c}$$



- and

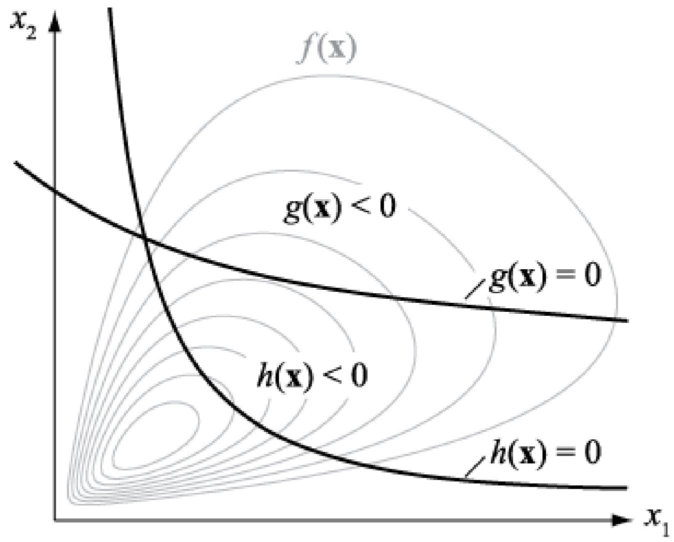
$$\Pr(Z|\boldsymbol{X} = \boldsymbol{x}) = \alpha L(\mathbf{x})$$
$$= \frac{\alpha}{c}\Pr[P \leq cL(\mathbf{x})]$$

it follows
$$\Pr(Z) = \int_{\mathbf{X}} \Pr(Z|\boldsymbol{X} = \boldsymbol{x})\, f(\mathbf{x}) \mathrm{d}\mathbf{x}$$
$$= \frac{\alpha}{c} \int_{\mathbf{X}} \Pr[P \leq cL(\mathbf{x})]\, f(\mathbf{x}) \mathrm{d}\mathbf{x}$$

the event $\{P \leq cL(\mathbf{x})\}$ is represented through the limit state function
$$h_e(\mathbf{x}, p) = p - cL(\mathbf{x})$$

and corresponding domain $\Omega_{Ze} = \{h_e(\mathbf{x}, p) \leq 0\}$

thus
$$\Pr(Z) = \frac{\alpha}{c} \int_{\mathbf{x}, p \in \Omega_{Ze}} f(p) f(\mathbf{x}) \mathrm{d}\mathbf{x}\, \mathrm{d}p$$
$$= \frac{\alpha}{c} \int_{\mathbf{x}, p \in \Omega_{Ze}} f(\mathbf{x}) \mathrm{d}\mathbf{x}\, \mathrm{d}p$$



accordingly $$\Pr(F \cap Z) = \int_{\mathbf{X}} \Pr(Z|\boldsymbol{X} = \boldsymbol{x}) \Pr(F|\boldsymbol{X} = \boldsymbol{x}) f(\mathbf{x}) \mathrm{d}\mathbf{x}$$

$$= \frac{\alpha}{c} \int_{\mathbf{x},p \in \{\Omega_F \cap \Omega_{Ze}\}} f(\mathbf{x}) \mathrm{d}\mathbf{x}\, \mathrm{d}p$$

and finally

$$\Pr(F|Z) = \frac{\Pr(F \cap Z)}{\Pr(Z)} = \frac{\int_{\mathbf{x},p \in \{\Omega_F \cap \Omega_{Ze}\}} f(\mathbf{x}) \mathrm{d}\mathbf{x}\, \mathrm{d}p}{\int_{\mathbf{x},p \in \Omega_{Ze}} f(\mathbf{x}) \mathrm{d}\mathbf{x}\, \mathrm{d}p}$$

Both terms can be solved by any Structural Reliability Method, since all domains are described by inequalities

Straub D. (2011). Reliability updating with equality information. *Probabilistic Engineering Mechanics*, **26**(2), pp. 254–258.

Application to spatially distributed systems – an exploratory example

- Observations of a Gaussian process $\mathbf{X} = [X_1, X_2, \dots, X_n]^{\mathrm{T}}$
- Failure at location *t*: $g_t(\mathbf{x}) = x_t - 2$
- Measurements:

Table 1. Measurements made of the process $X(t)$ at different locations t_m.

$t_{m,i} =$	10	20	30	40	50	60	70	80	90
$x_{m,i} =$	5	3	4	6	5	5	3	3	4

Demonstration example:
Updating of a Gaussian process with 9 measurements

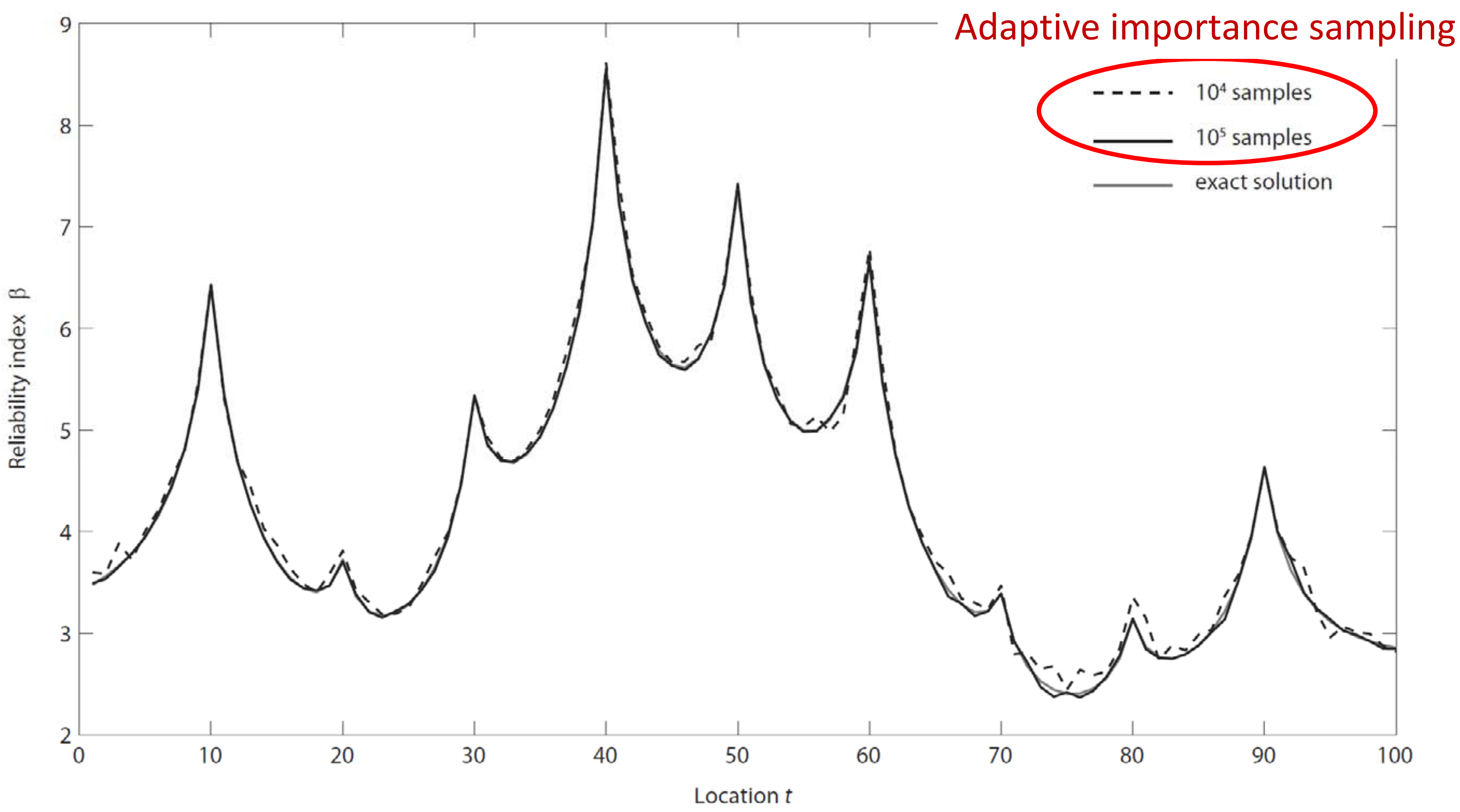


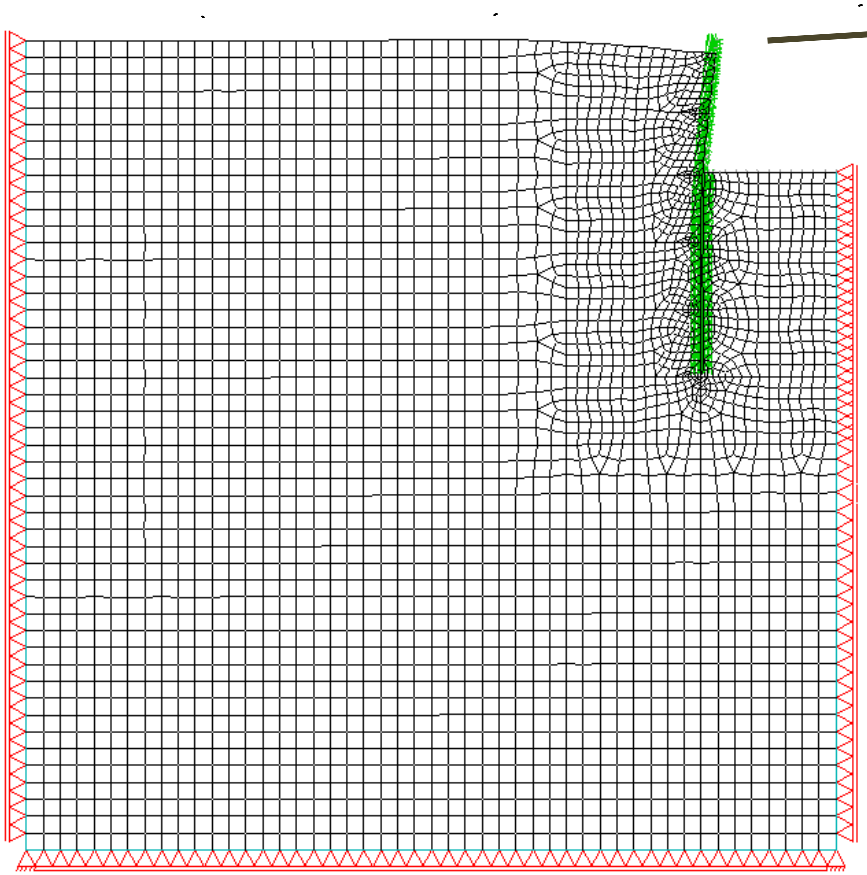
Figure 6. Reliability index at locations $\mathbf{t} = [1; 2; \dots; 100]$, conditional on measurements of the process $\mathbf{X}(t)$ at locations $t = [10; 20; \dots; 90]$.

Application 2:
How to compute the reliability of a geotechnical site conditional on monitoring?

- Papaioannou I., Straub D. (2012). *Computers & Geotechnics*, **42**: 44–51.

Reliability updating during construction

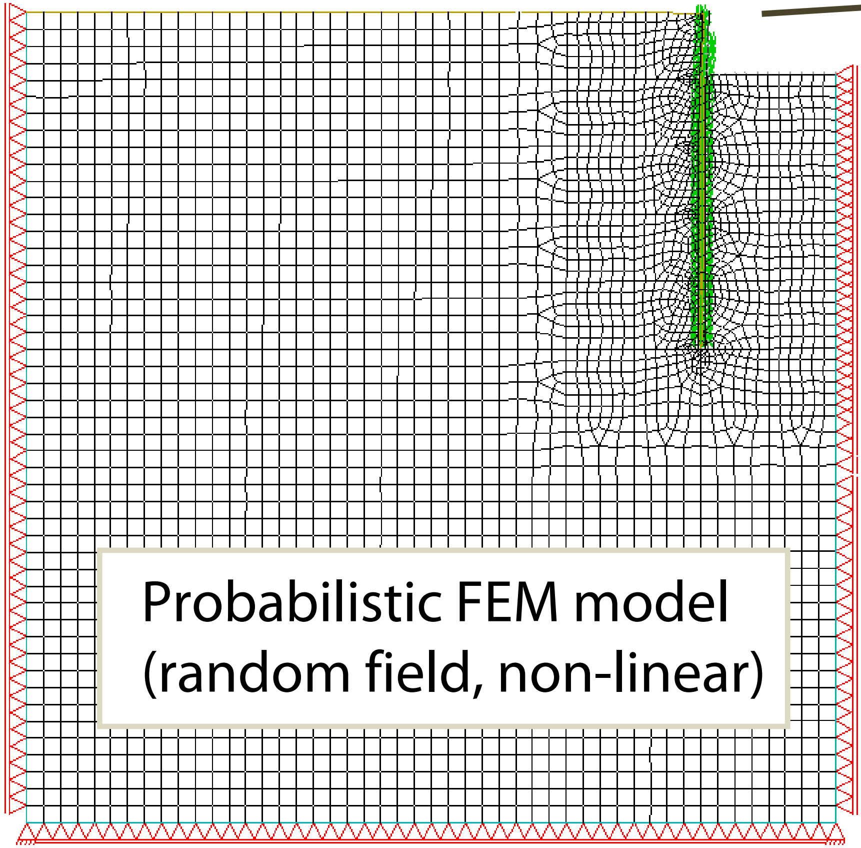
Example: Geotechnical site



Deformation should be limited

Reliability updating during construction

Example: Geotechnical site

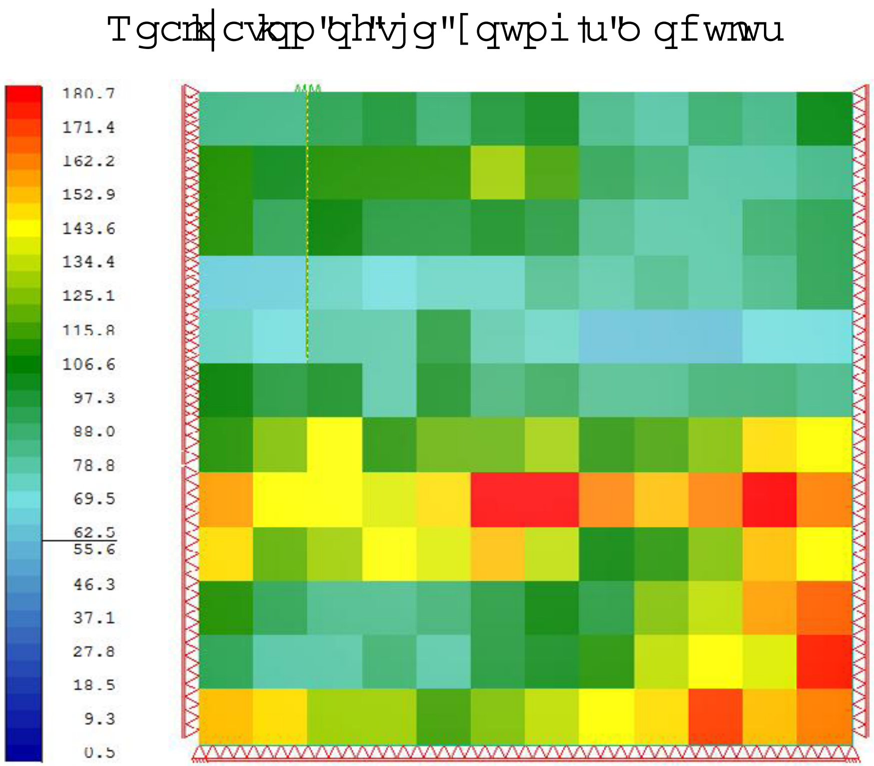


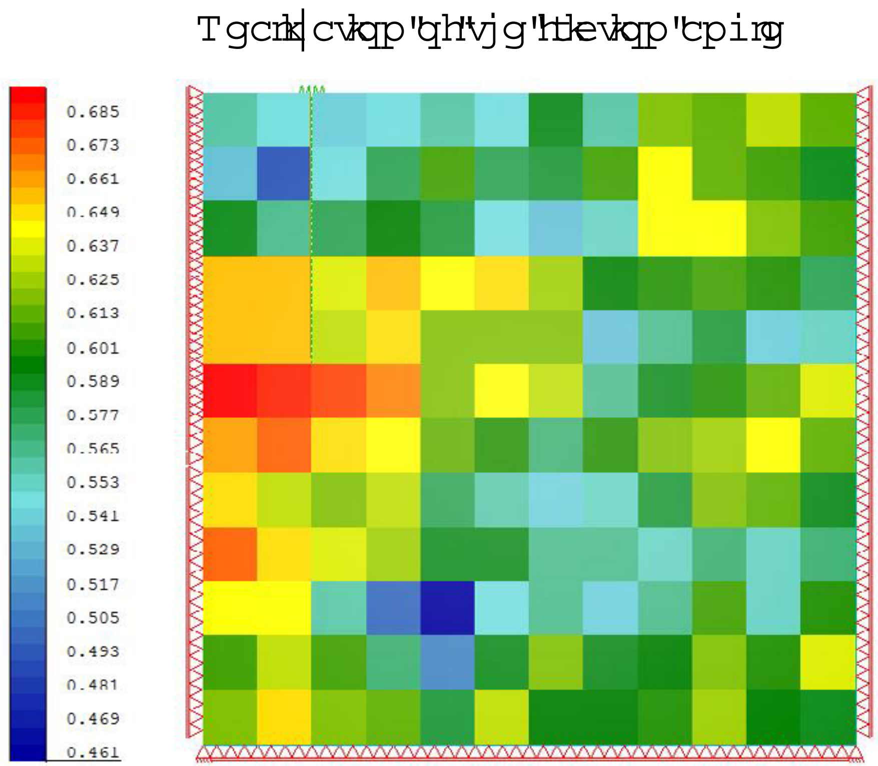
Deformation
at intermediate stage
can be measured

Probability (of the final
stage) is updated

Random field realizations

(Homogenous, anisotropic random field)





Solution strategy

FEM model

- Limit state function for failure:

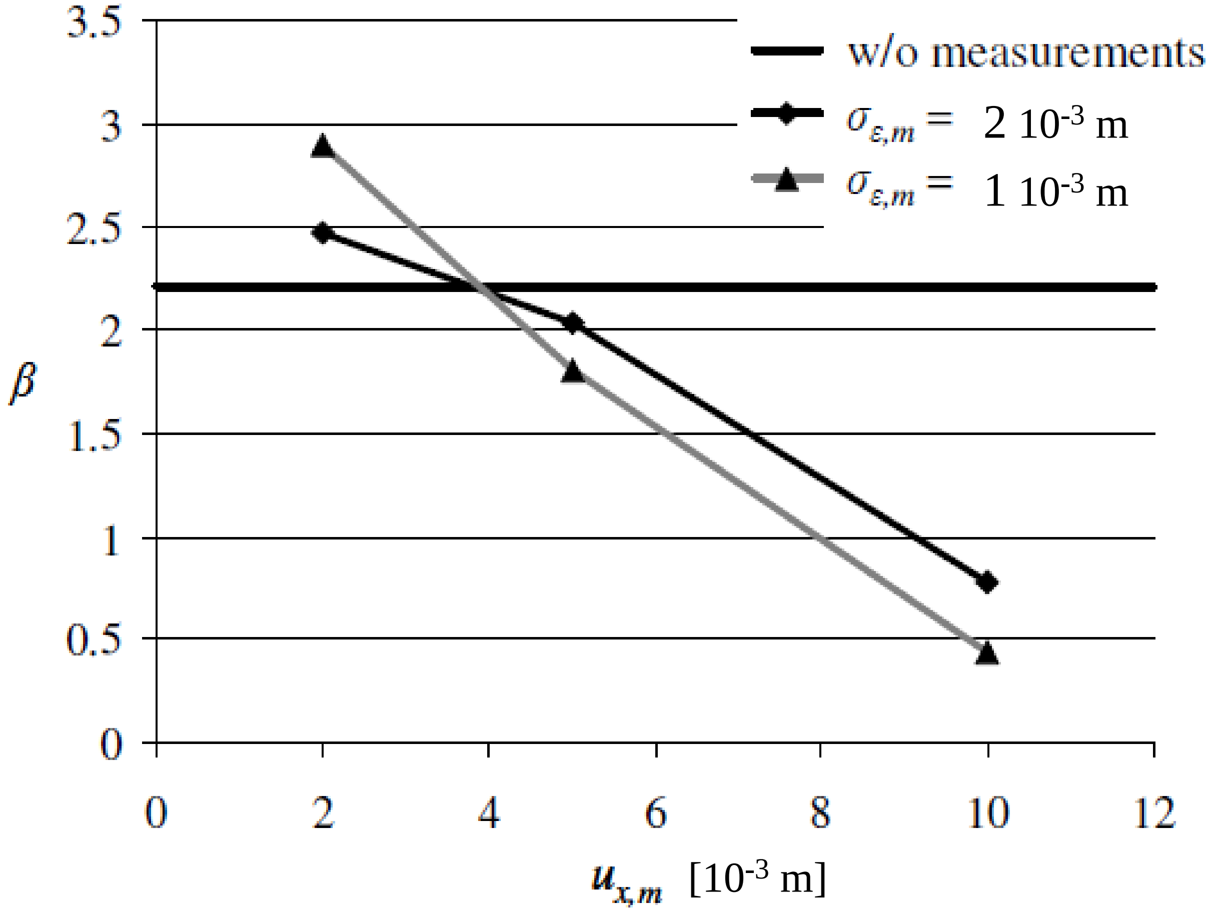
$$g(\mathbf{X}) = 0.1\mathrm{m} - u(\mathbf{X})$$

- Limit state function for observation:

$$h_e(\mathbf{X}) = p - \frac{c}{\sigma_\epsilon}\varphi\left(\frac{m - u_m(\mathbf{X})}{\sigma_\epsilon}\right)$$

- Evaluated with Subset simulation
  ($2 - 3 \cdot 10^3$ LSF calls)

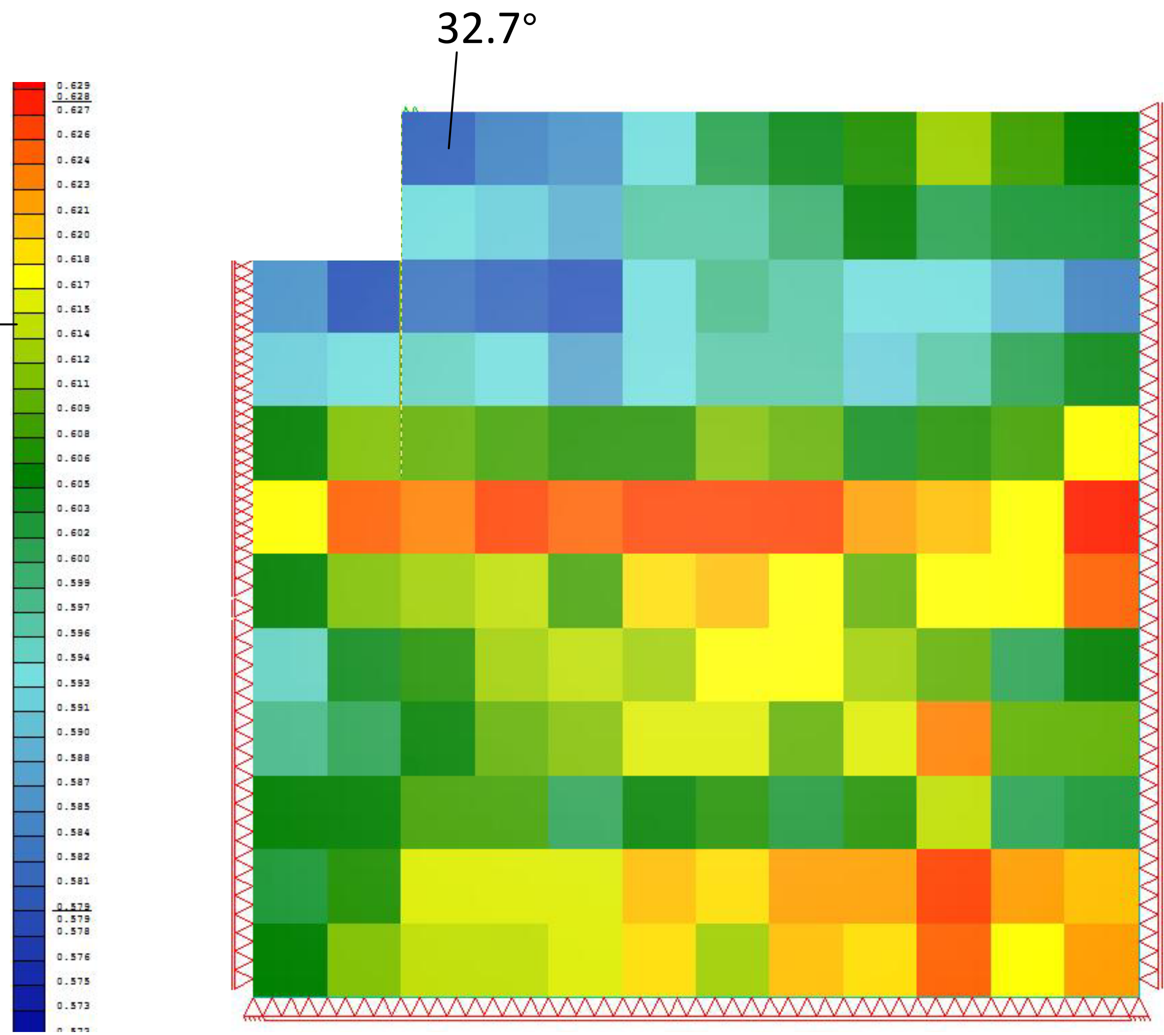
Reliability index vs measured displacements



Updated random soil parameters

Mean of friction angle

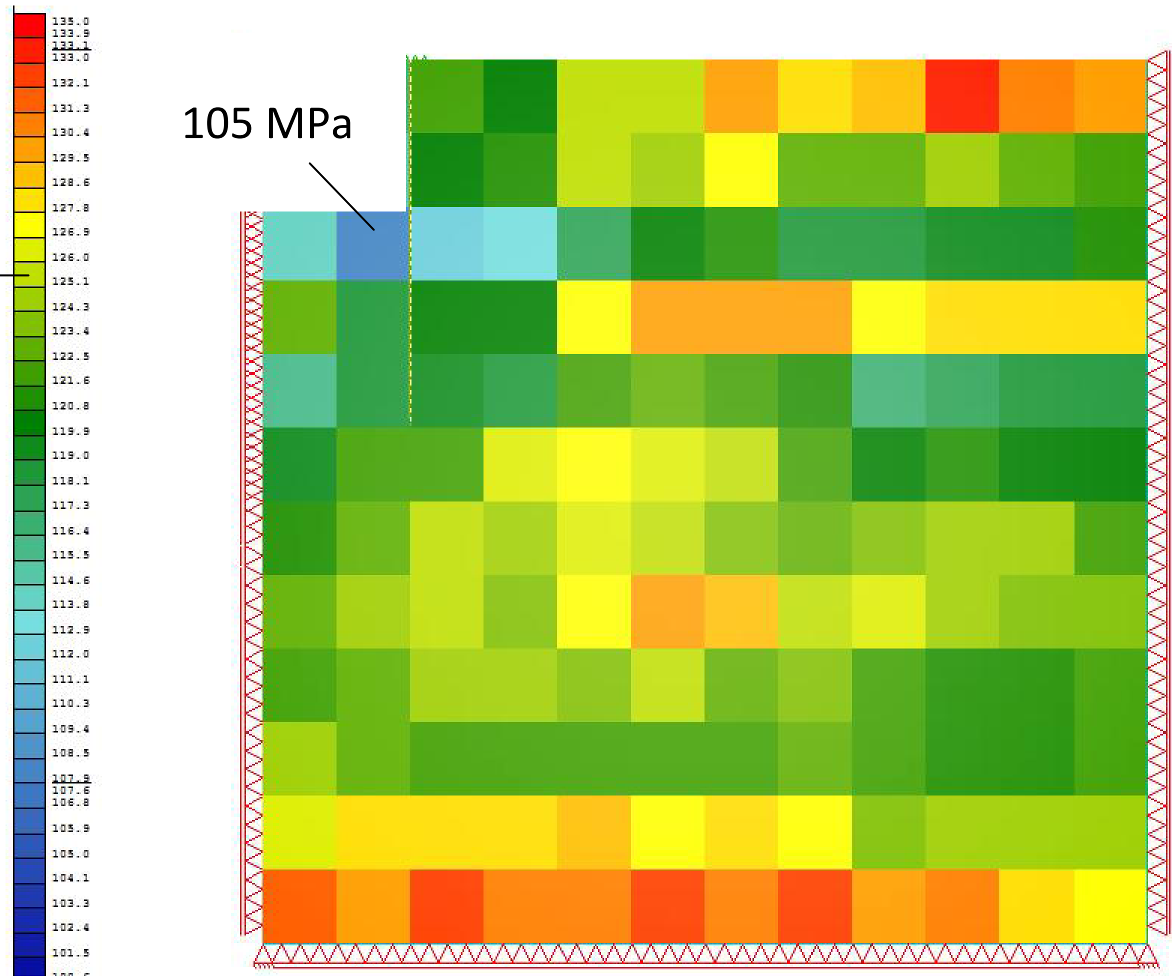
- Prior mean: 35°



Updated random soil parameters

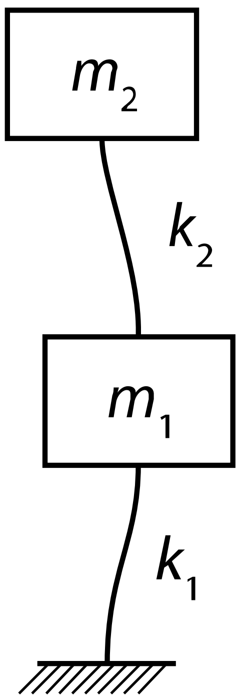
Young‘s modulus E

- Prior mean: 125 MPa



Dynamical system identification

(from Beck and Au 2002)



- Idealized model with two uncertain parameters $k_1 = X_1 k$ and $k_2 = X_2 k$

- Measurements of the first two Eigenfrequencies of the real system $f_1^m = 3.13\text{Hz}$ and $f_2^m = 9.83\text{Hz}$

- Likelihood function:

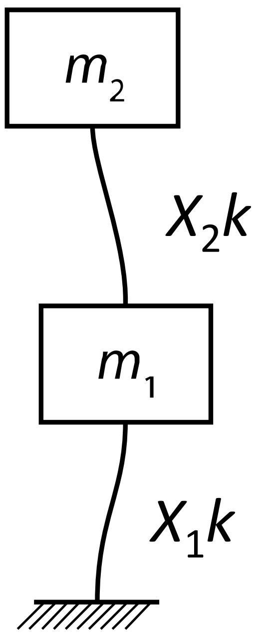
$$L(\mathbf{x}) = f_\epsilon(f_1^m - f_1(\mathbf{x}))f_\epsilon(f_2^m - f_2(\mathbf{x}))$$

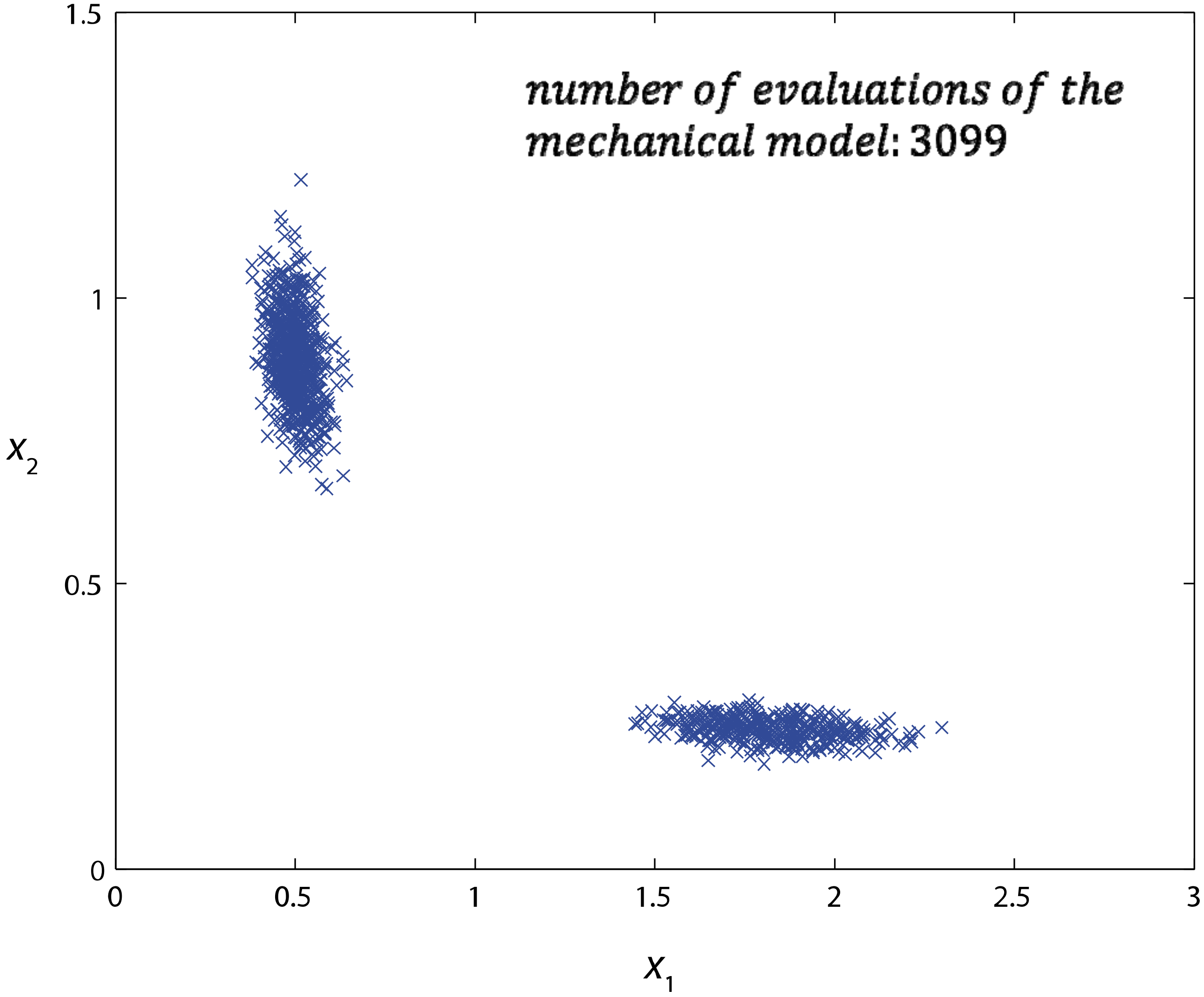
- with model results $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ from

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} kx_1 + kx_2 & -kx_2 \\ -kx_2 & kx_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

Bayesian parameter identification

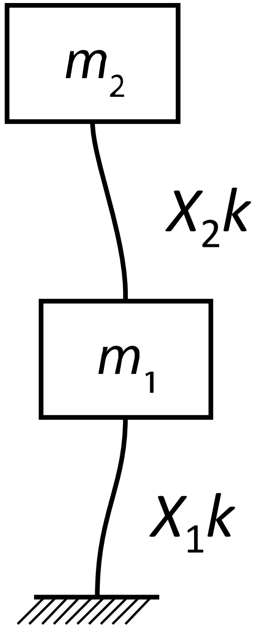
Simulation results

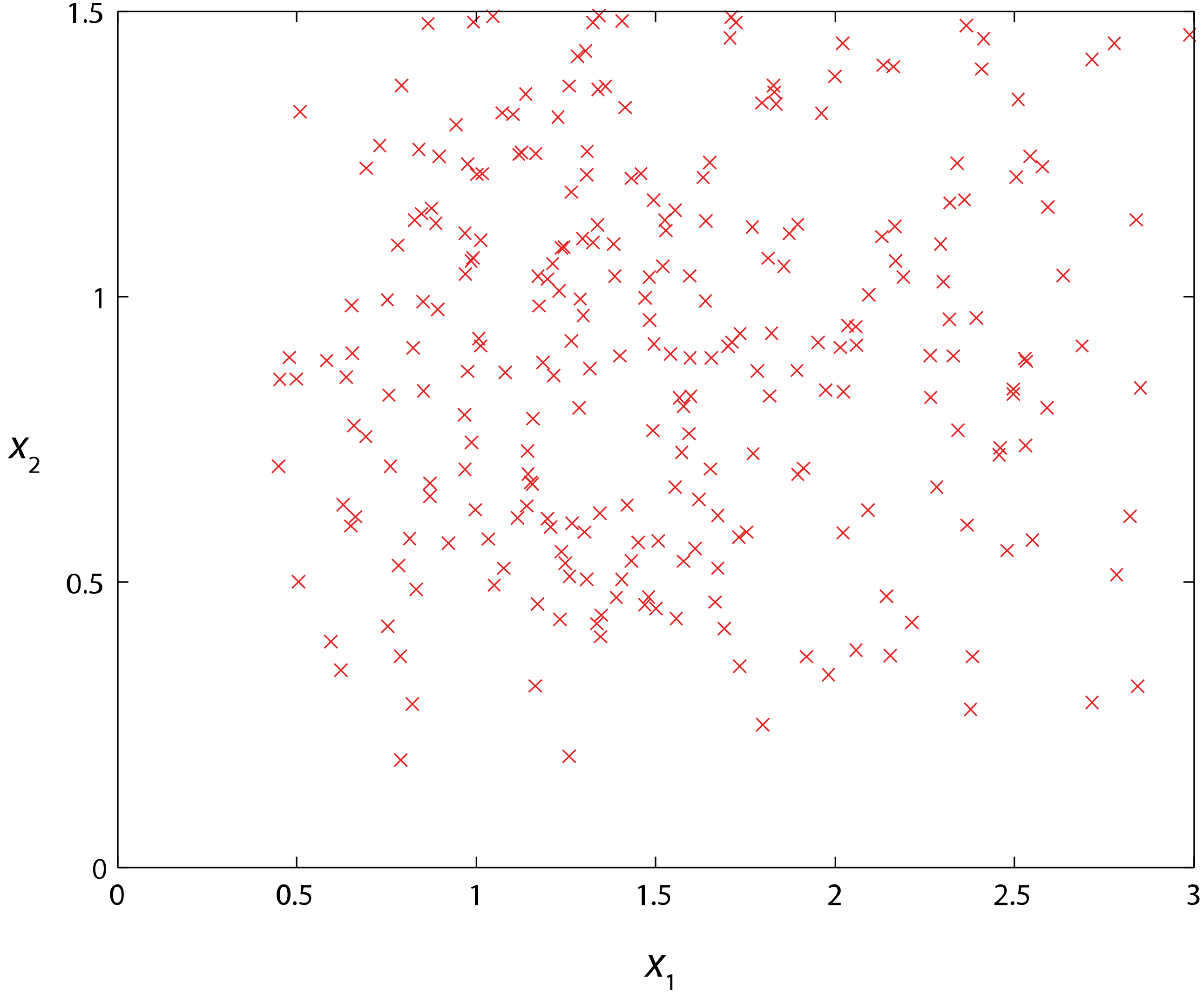




Simulation algorithm

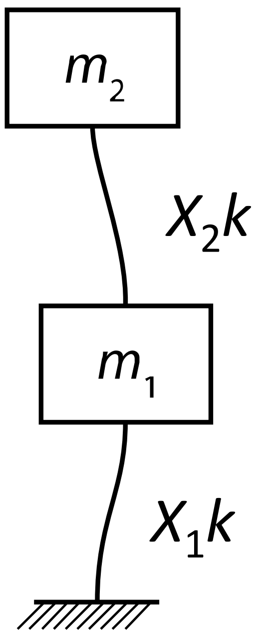
Subset simulation level 1

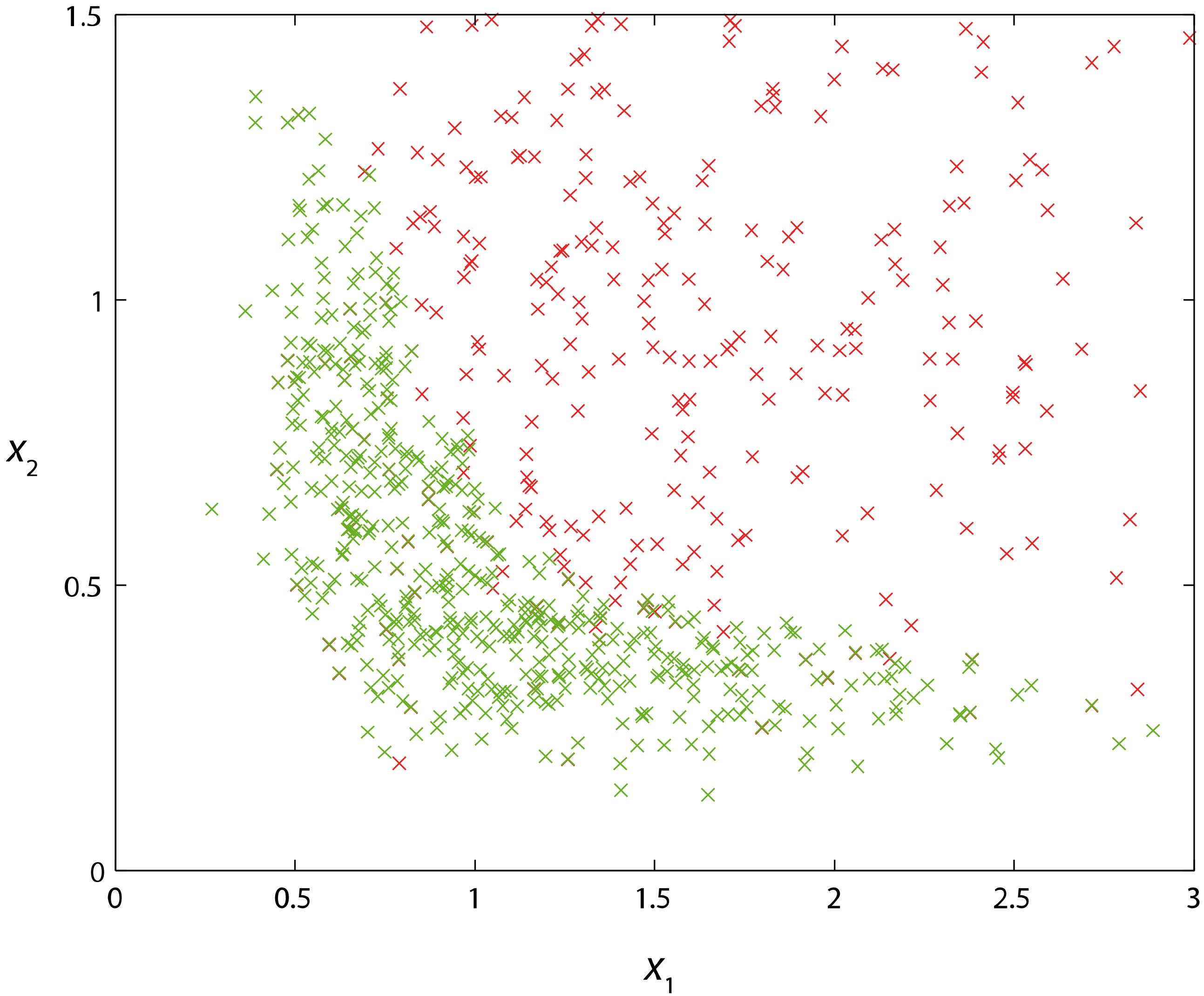




Simulation algorithm

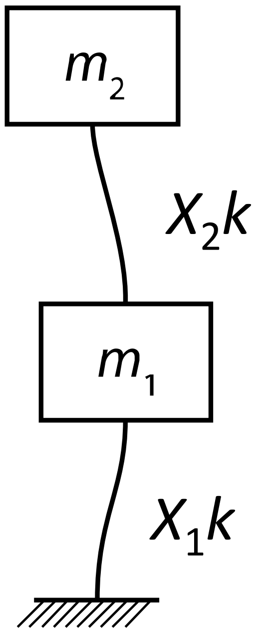
Subset simulation level 2

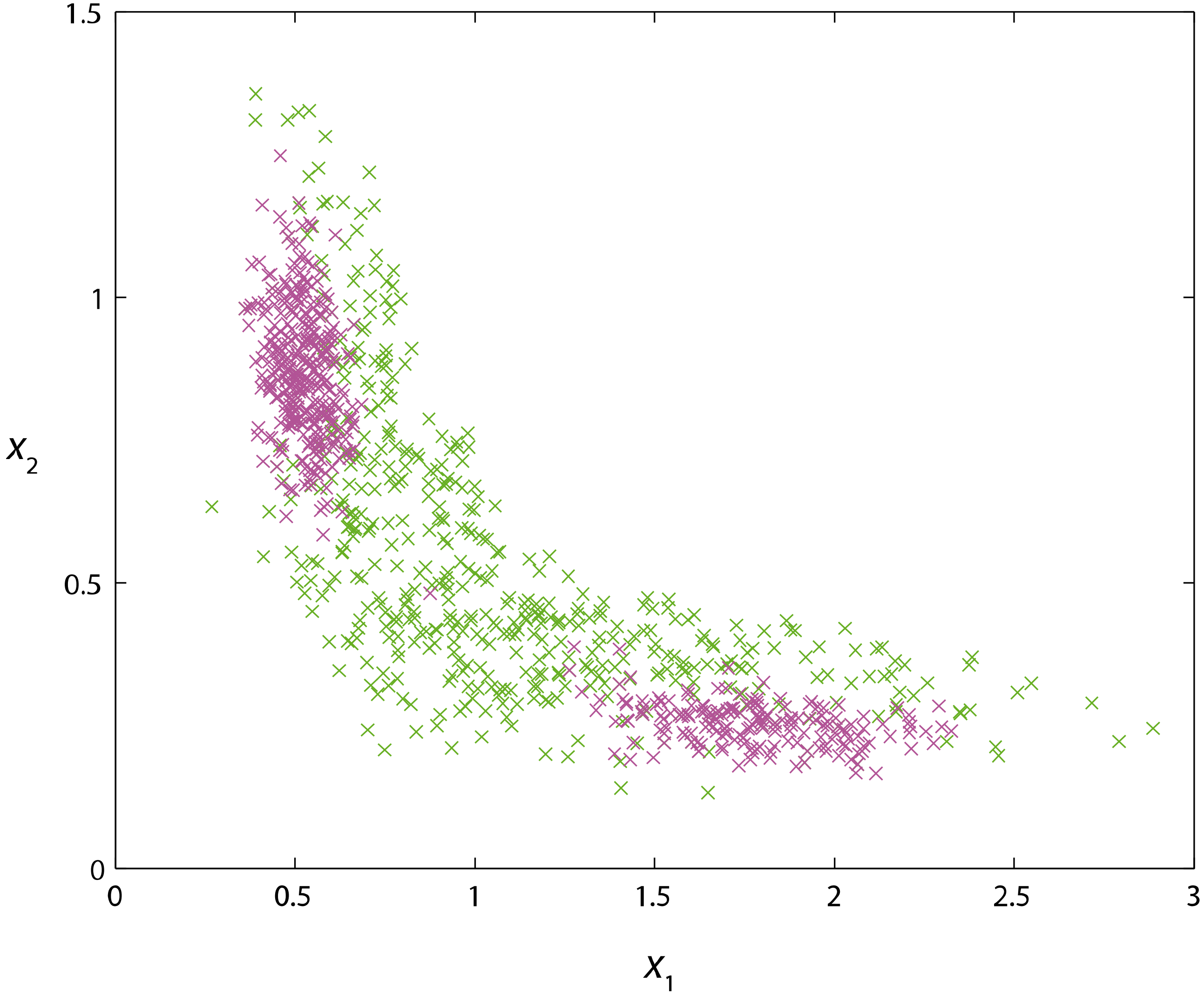




Simulation algorithm

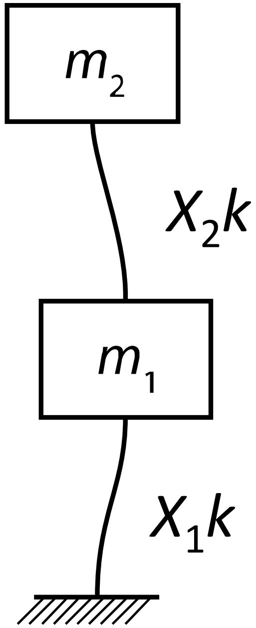
Subset simulation level 3

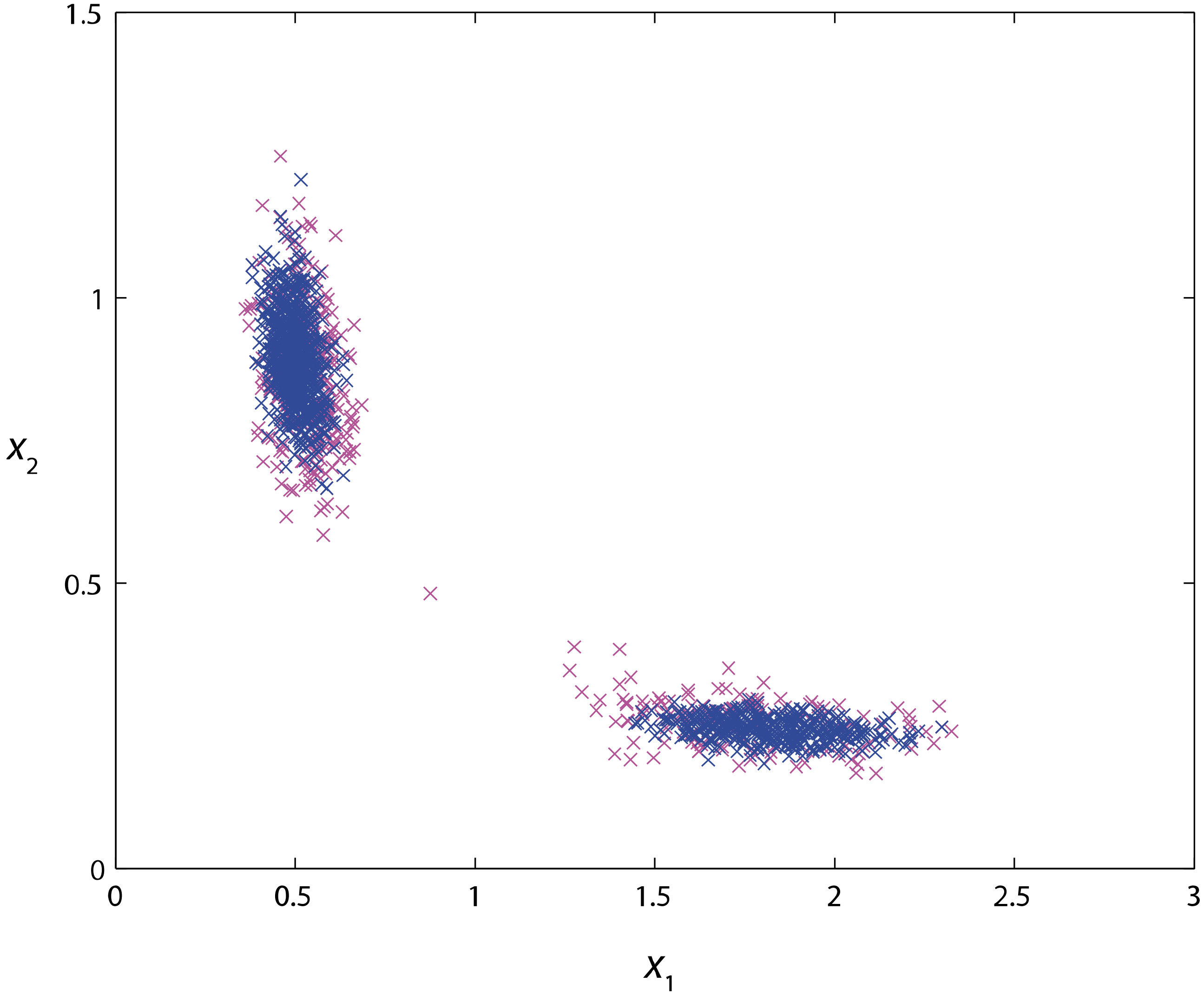




Simulation algorithm

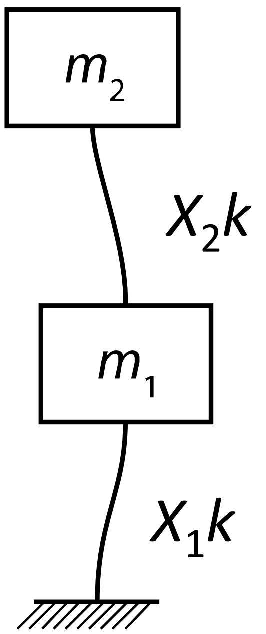
Subset simulation level 4: final samples

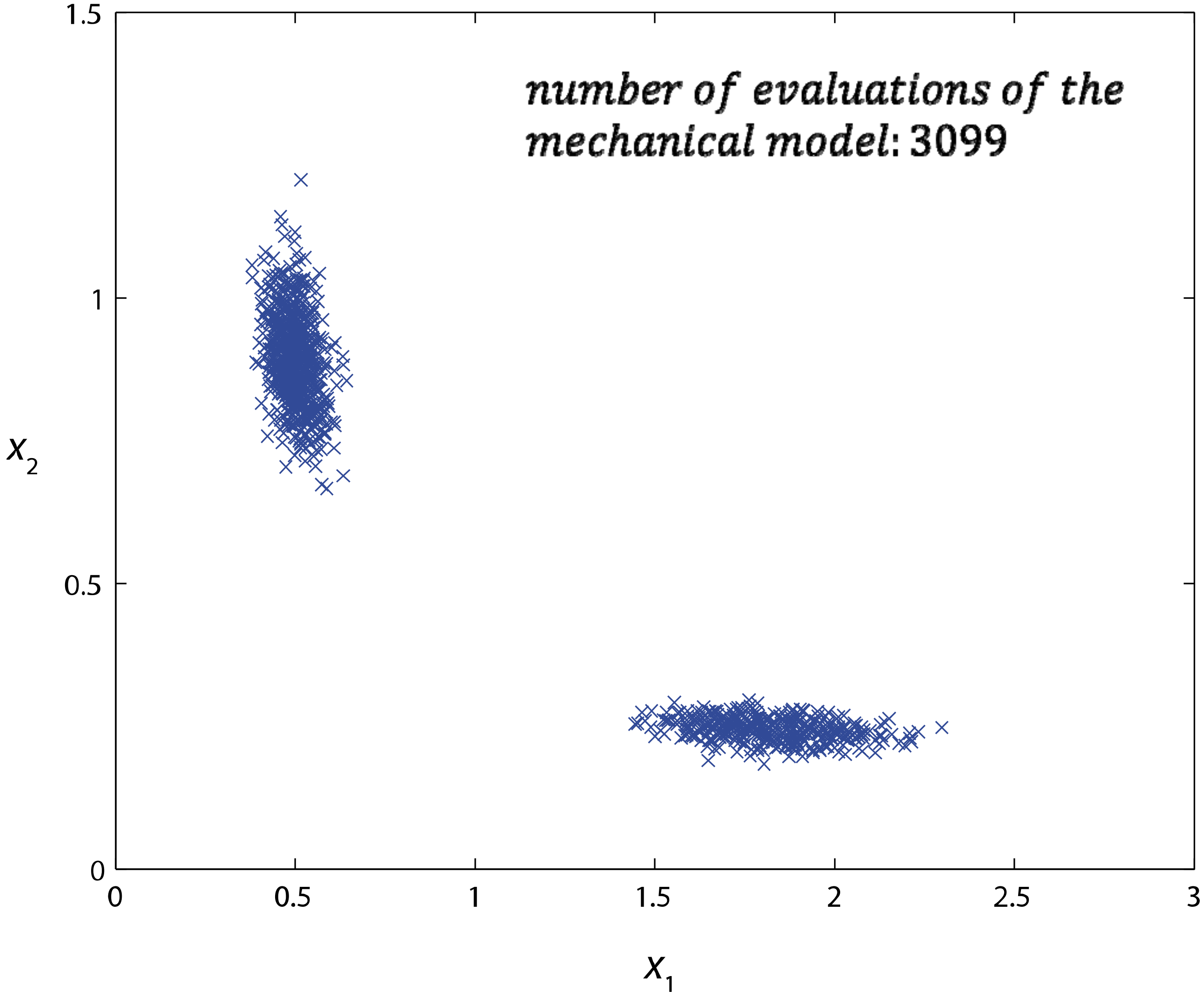




Simulation algorithm

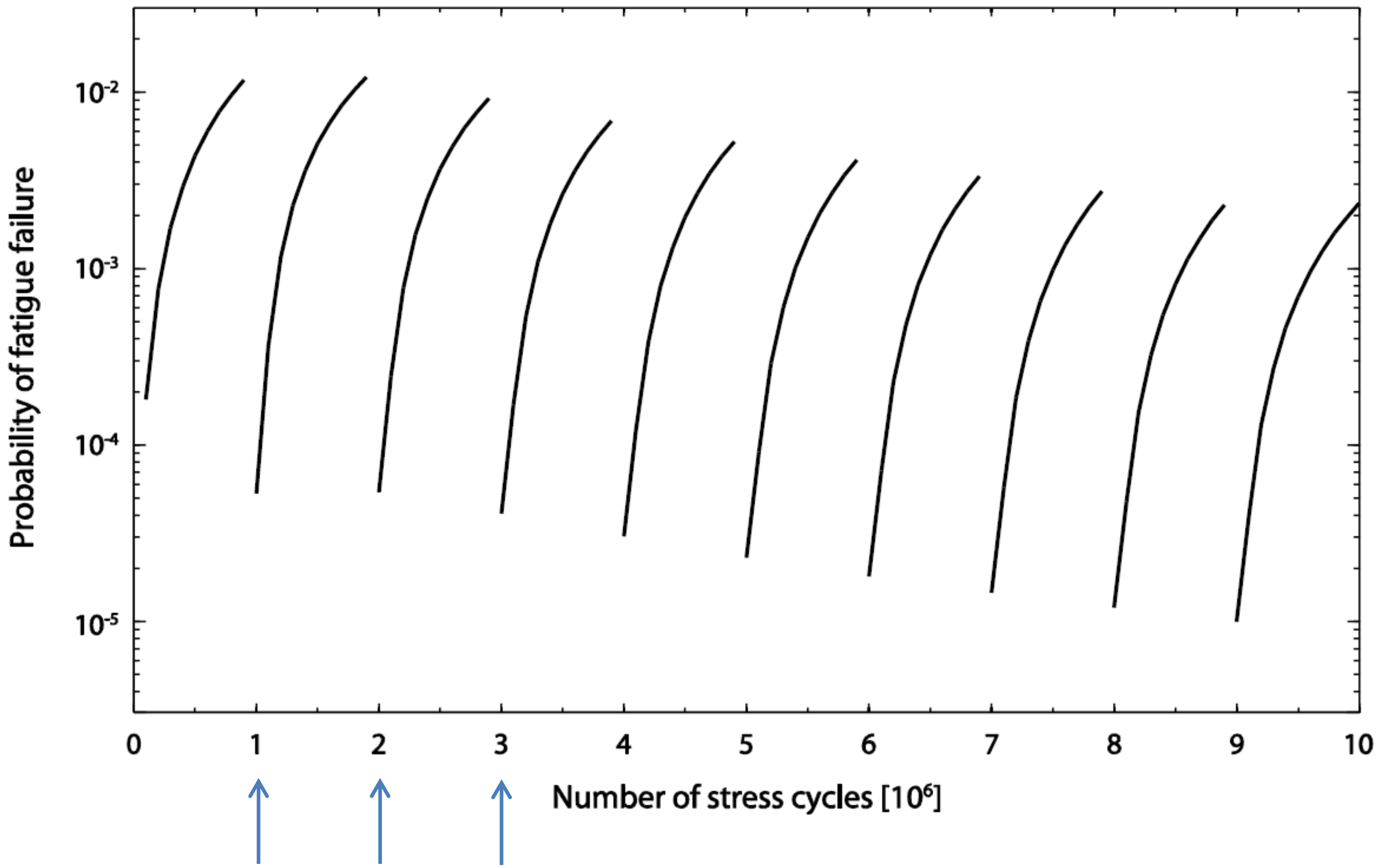
Subset simulation level 4: final samples





Include all information in near-real-time over the lifetime

Example: fatigue reliability



Inspections

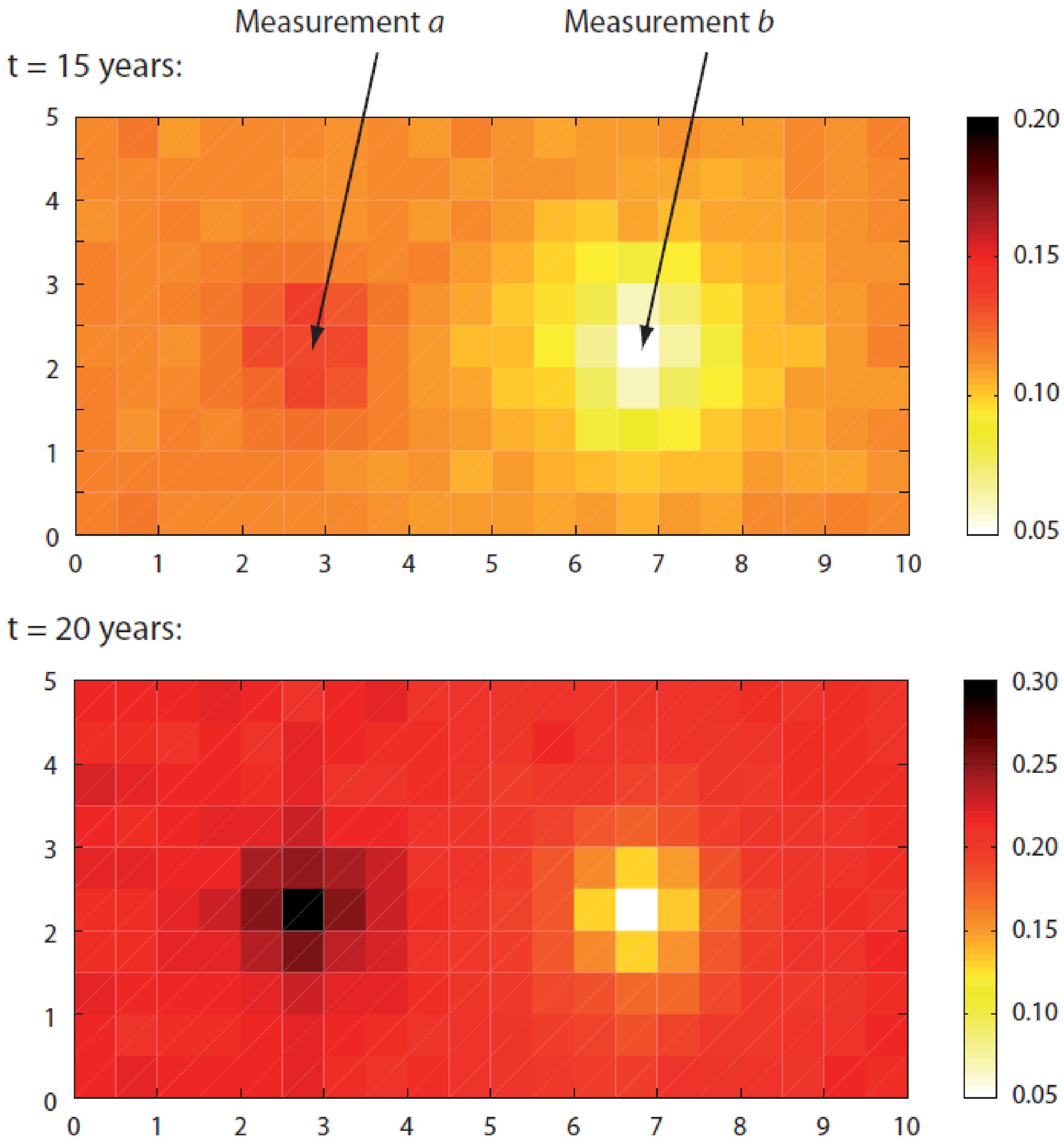
Straub D. (2009). *ASCE J Engineering Mechanics*

Application: Corrosion of reinforcement in concrete

- Corrosion caused by ingress of chlorides
- Chloride profile measurements

Straub D. & Fischer J. (2011) *Proc. ICASP*

Influence of chloride measurements on probability of corrosion



Potential in mechanical and civil engineering is huge

Additional past/current projects on Bayesian updating

Updating probabilistic models with observations for:

- Acoustic emission (Schumacher & Straub 2011)
- Avalanche risk (Straub & Grêt-Regamey 2006)
- Flood damage assessment (Frey, Butenuth & Straub 2012)
- SHM of aircraft structures (EU project ROSA)
- Structural systems (Straub & Der Kiureghian 2010)
- Tunnel construction (Špačková & Straub in print)
- Aging ship structures (Luque & Straub, in preparation)

To conclude…

- Bayesian updating enables to include any relevant observation into your prediction
- Presented methods based on structural reliability are efficient and simple
- Simple (robust) importance sampling and subset simulation schemes perform well for updating in large systems

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